Sound Bisimulations for Higher-Order Distributed Process Calculus*

Adrien Piérard and Eijiro Sumii**

Tohoku University
{adrien,sumii}@kb.ecei.tohoku.ac.jp

Abstract. While distributed systems with transfer of processes have become pervasive, methods for reasoning about their behaviour are underdeveloped. In this paper we propose a bisimulation technique for proving behavioural equivalence of such systems modelled in the *higher-order π-calculus with passivation* (and restriction). Previous research for this calculus is limited to context bisimulations and normal bisimulations which are either impractical or unsound. In contrast, we provide a sound and useful definition of *environmental bisimulations*, with several non-trivial examples. Technically, a central point in our bisimulations is the clause for parallel composition, which must account for passivation of the spawned processes in the middle of their execution.

1 Introduction

1.1 Background

Higher-order distributed systems are ubiquitous in today's computing environment. To name but a few examples, companies like Dell and Hewlett-Packard sell products using virtual machine live migration [14, 3], and Gmail users execute remote JavaScript code on local browsers. In this paper we call *higher-order* the ability to transfer processes, and *distribution* the possibility of location-dependent system behaviour. In spite of the *de facto* importance of such systems, they are hard to analyse because of their inherent complexity.

The π-calculus [8] and its dialects prevail as models of concurrency, and several variations of these calculi have been designed for distribution. First-order variations include the ambient calculus [1] and Dπ [2], while higher-order include more recent Homer [4] and Kell [15] calculi. In this paper, we focus on the higher-order π-calculus with *passivation* [7], a simple high-level construct to express distribution. It is an extension of the higher-order π-calculus [9] (with which the reader is assumed to be familiar) with *located processes* $a[P]$ and two additional transition rules: $a[P] \xrightarrow{\overline{a}\langle P\rangle} 0$ (PASSIV), and $a[P] \xrightarrow{\alpha} a[P']$ if $P \xrightarrow{\alpha} P'$ (TRANSP).

* Appendix with full proofs at http://www.kb.ecei.tohoku.ac.jp/~adrien/pubs/SoundAppendix.pdf

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The new syntax $a[P]$ reads as "process P located at a" where a is a name. Rule TRANSP specifies the transparency of locations, i.e. that a location has no impact on the transitions of the located process. Rule PASSIV indicates that a located process can be *passivated*, that is, be output to a channel of the same name as the location. Using passivation, various characteristics of distributed systems are expressible. For instance, failure of process P located at a can be modelled like $a[P] \mid a(X).\overline{fail} \to 0 \mid \overline{fail}$, and migration of process Q from location b to c like $b[P] \mid b(X).c[X] \to 0 \mid c[P]$.

One way to analyse the behaviour of systems is to compare implementations and specifications. Such comparison calls for satisfying notions of behavioural equivalence, such as *reduction-closed barbed equivalence* (and *congruence*) [5], written $\approx$ (and $\approx_c$ respectively) in this paper.

Unfortunately, these equivalences have succinct definitions that are not very practical as a proof technique, for they both include a condition that quantifies over arbitrary processes, like: if $P \approx Q$ then $\forall R.\ P \mid R \approx Q \mid R$. Therefore, more convenient definitions like *bisimulations*, for which membership implies behavioural equivalence, and which come with a co-inductive proof method, are sought after.

Still, the combination of both higher order and distribution has long been considered difficult. Recent research on higher-order process calculi led to defining sound *context bisimulations* [10] (often at the cost of appealing to Howe's method [6] for proving congruence) but those bisimulations suffer from their heavy use of universal quantification: suppose that $\nu\widetilde{c}.\overline{a}\langle M\rangle.P\ \mathcal{X}\ \nu\widetilde{d}.\overline{a}\langle N\rangle.Q$, where $\mathcal{X}$ is a context bisimulation; then it is roughly required that for any process R, we have $\nu\widetilde{c}.(P \mid R\{M/X\})\ \mathcal{X}\ \nu\widetilde{d}.(Q \mid R\{N/X\})$. Not only must we consider the outputs M and N, but we must also handle interactions of arbitrary R with the continuation processes P and Q. Alas, this almost comes down to showing reduction-closed barbed equivalence! In the higher-order π-calculus, by means of encoding into a first-order calculus, normal bisimulations [10] coincide with (and are a practical alternative to) context bisimulations. Unfortunately, normal bisimulations have proved to be unsound in the presence of passivation (and restriction) [7]. While this result cast a doubt on whether sound normal bisimulations exist for higher-order distributed calculi, it did not affect the potential of environmental bisimulations [16, 17, 12, 13] as a useful proof technique for behavioural equivalence in those calculi.

1.2 Our contribution

To the best of our knowledge, there are not yet any useful sound bisimulations for higher-order distributed process calculi. In this paper we develop environmental (weak) bisimulations for the higher-order π-calculus with passivation, which (1) are sound with respect to reduction-closed barbed equivalence, (2) can actually be used to prove behavioural equivalence of non-trivial processes (with restrictions), and (3) can also be used to prove reduction-closed barbed *congruence* of processes (see Corollary 1). To prove reduction-closed barbed equivalence (and congruence), we find a new clause to guarantee preservation of bisimilarity by parallel composition of arbitrary processes. Unlike the corresponding clause in previous research [7, 13], it can also handle the later removal (i.e. passivation) of these processes while keeping the bisimulation proofs tractable. Several examples are given, thereby supporting our claim of the first useful bisimulations for a higher-order distributed process calculus. Moreover, we define an up-to context variant of the environmental bisimulations that significantly lightens the burden of equivalence proofs, as utilised in the examples.

Overview of the bisimulation We now outline the definition of our environmental bisimulations. (Generalities on environmental bisimulations can be found in [12].) We define an environmental bisimulation $\mathcal{X}$ as a set of quadruples $(r, \mathcal{E}, P, Q)$ where r is a set of names (i.e. channels and locations), $\mathcal{E}$ is a binary relation (called the *environment*) on terms, and P, Q are processes. The bisimulation is a game where the processes P and Q are compared to each other by an *attacker* (or *observer*) who knows and can use the terms in the environment $\mathcal{E}$ and the names in r. For readability, the membership $(r, \mathcal{E}, P, Q) \in \mathcal{X}$ is often written $P \; \mathcal{X}_{\mathcal{E};r} \; Q$, and should be understood as "processes P and Q are bisimilar, under the environment $\mathcal{E}$ and the known names r."

The environmental bisimilarity is co-inductively defined by several conditions concerning the tested processes and the knowledge. As usual with weak bisimulations, we require that an internal transition by one of the processes is matched by zero or more internal transitions by the other, and that the remnants are still bisimilar.

As usual with (more recent and less common) environmental bisimulations, we require that whenever a term M is output to a known channel, the other tested process can output another term N to the same channel, and that the residues are bisimilar under the environment extended with the pair (M, N). The extension of the environment stands for the growth of knowledge of the attacker of the bisimulation game who observed the outputs (M, N), although he cannot analyse them. This spells out like: for any $P \; \mathcal{X}_{\mathcal{E};r} \; Q$ and $a \in r$, if $P \xrightarrow{\nu\widetilde{c}.\overline{a}\langle M\rangle} P'$ for fresh $\widetilde{c}$, then $Q \overset{\nu\widetilde{d}.\overline{a}\langle N\rangle}{\Longrightarrow} Q'$ for fresh $\widetilde{d}$ and $P' \; \mathcal{X}_{\mathcal{E}\cup\{(M,N)\};r} \; Q'$.

Unsurprisingly, input must be doable on the same known channel by each process, and the continuations must still be bisimilar under the same environment since nothing is learnt by the context. However, we require that the input terms are generated from the *context closure* of the environment. Intuitively, this closure represents all the processes an attacker can build by combining what he has learnt from previous outputs. Roughly, we define it as:

$$(\mathcal{E}; r)^\star = \{(C[\widetilde{M}], C[\widetilde{N}]) \mid C \; context, \; fn(C) \subseteq r, \; \widetilde{M} \; \mathcal{E} \; \widetilde{N}\}$$

where $\widetilde{M}$ denotes a sequence $M_0, \ldots, M_n$, and $\widetilde{M} \mathcal{E} \widetilde{N}$ means that for all $0 \leq i \leq n$, $M_i \mathcal{E} N_i$. Therefore, the input clause looks like: for any $P \; \mathcal{X}_{\mathcal{E};r} \; Q$, $a \in r$ and $(M, N) \in (\mathcal{E}; r)^\star$, if $P \xrightarrow{a(M)} P'$, then $Q \overset{a(N)}{\Longrightarrow} Q'$ and $P' \; \mathcal{X}_{\mathcal{E};r} \; Q'$.

The set r of known names can be extended at will by the observer, provided that the new names are fresh: for any $P \; \mathcal{X}_{\mathcal{E};r} \; Q$ and n fresh, we have $P \; \mathcal{X}_{\mathcal{E};r\cup\{n\}} \; Q$.

Parallel composition The last clause is crucial to the soundness and usefulness of environmental bisimulations for languages with passivation, and not as straightforward as the other clauses. The idea at its base is that not only may an observer run arbitrary processes R in parallel to the tested ones (as in reduction-closed barbed equivalence), but he may also run arbitrary processes M, N he assembled from previous observations. It is critical to ensure that bisimilarity (and hopefully equivalence) is preserved by such parallel composition, and that this property can be easily proved. As $(\mathcal{E}; r)^\star$ is this set of processes that can be assembled from previous observations, we would naively expect the appropriate clause to look like:

$$\text{For any } P \; \mathcal{X}_{\mathcal{E};r} \; Q \text{ and } (M, N) \in (\mathcal{E}; r)^\star, \text{ we have } P \mid M \; \mathcal{X}_{\mathcal{E};r} \; Q \mid N$$

but this subsumes the already impractical clause of reduction-closed barbed equivalence which we want to get round. Previous research [7, 13] uses a weaker condition:

$$\text{For any } P \; \mathcal{X}_{\mathcal{E};r} \; Q \text{ and } (M, N) \in \mathcal{E}, \text{ we have } P \mid M \; \mathcal{X}_{\mathcal{E};r} \; Q \mid N$$

arguing that $(\mathcal{E}; r)^\star$ can informally do no more observations than $\mathcal{E}$, but this clause is unsound in the presence of passivation. The reason behind the unsoundness is that, in our settings, not only can a context spawn new processes M, N, but it can also *remove* running processes it created by passivating them later on. For example, consider the following processes $P = \overline{a}\langle R\rangle.!R$ and $Q = \overline{a}\langle 0\rangle.!R$. Under the above weak condition, it would be easy to construct an environmental bisimulation that relates P and Q. However, a process $a(X).m[X]$ may distinguish them. Indeed, it may receive processes R and start running it in location m, or may receive process 0 and run a copy of R from $!R$. If R is a process doing several sequential actions (for example if $R = \mathit{lock}.\mathit{unlock}$) and is passivated *in the middle* of its execution, then the remaining processes after passivation would not be equivalent any more.

To account for this new situation, we decide to modify the condition on the provenance of process that can be spawned, drawing them from $\{(a[M], a[N]) \mid a \in r, \; (M, N) \in \mathcal{E}\}$, thus giving the clause:

$$\text{For any } P \; \mathcal{X}_{\mathcal{E};r} \; Q, \; a \in r \text{ and } (M, N) \in \mathcal{E}, \text{ we have } P \mid a[M] \; \mathcal{X}_{\mathcal{E};r} \; Q \mid a[N].$$

The new condition allows for any running process that has been previously created by the observer to be passivated, that is, removed from the current test. This clause is much more tractable than the first one using $(\mathcal{E}; r)^\star$ and, unlike the second one using only $\mathcal{E}$, leads to sound environmental bisimulations (albeit with a limitation; see Remark 1).

Example With our environmental bisimulations, non-trivial equivalence of higher-order distributed processes can be shown, such as $P_0 = !a[e \mid \overline{e}]$ and $Q_0 = !a[e] \mid !a[\overline{e}]$, where e abbreviates $e(X).0$ and $\overline{e}$ is $\overline{e}\langle 0\rangle.0$. We explain here informally how we build a bisimulation $\mathcal{X}$ relating those processes.

$$\begin{aligned}\mathcal{X} = \{(r, \mathcal{E}, P, Q) \mid {} & r \supseteq \{a, e\}, \;\; \mathcal{E} = \{0, e, \overline{e}, e \mid \overline{e}\} \times \{0, e, \overline{e}\}, \\ & P \equiv P_0 \mid \textstyle\prod_{i=1}^n l_i[M_i], \;\; Q \equiv Q_0 \mid \prod_{i=1}^n l_i[N_i], \;\; n \geq 0, \\ & \widetilde{l} \in r, \;\; (\widetilde{M}, \widetilde{N}) \in \mathcal{E}\}\end{aligned}$$

Since we want $P_0 \; \mathcal{X}_{\mathcal{E};r} \; Q_0$, the spawning clause of the bisimulation requires that for any $(M_1, N_1) \in \mathcal{E}$ and $l_1 \in r$, we have $P_0 \mid l_1[M_1] \; \mathcal{X}_{\mathcal{E};r} \; Q_0 \mid l_1[N_1]$. Then, by repeatedly applying this clause, we obtain $(P_0 \mid \prod_{i=1}^n l_i[M_i]) \; \mathcal{X}_{\mathcal{E};r} \; (Q_0 \mid \prod_{i=1}^n l_i[N_i])$. Since the observer can add fresh names at will, we require r to be a superset of the free names $\{a, e\}$ of P_0 and Q_0. Also, we have the intuition that the only possible outputs from P and Q are processes $e \mid \overline{e}$, e, $\overline{e}$, and 0. Thus, we set ahead $\mathcal{E}$ as the Cartesian product of $\{0, e, \overline{e}, e \mid \overline{e}\}$ with $\{0, e, \overline{e}\}$, that is, the combination of expectable outputs. We emphasize that it is indeed reasonable to relate $\overline{e}, e$ and $e \mid \overline{e}$ to $0, e$ and $\overline{e}$ in $\mathcal{E}$ for the observer

$$P_0 \mid \prod_{i=1}^{n} l_i[M_i] \xrightarrow{\gamma \in \{e, \overline{e}\}} P_0 \mid a[\overline{\gamma}] \mid \prod_{i=1}^{n} l_i[M_i] \quad \text{(i)}$$
$$= P_0 \mid \prod_{i=1}^{n+1} l_i[M_i] \quad \text{for } l_{n+1} = a,\ M_{n+1} = \overline{\gamma}$$
$$P_0 \mid \prod_{i=1}^{n} l_i[M_i] \xrightarrow{\overline{l_n}\langle M_n \rangle} P_0 \mid \prod_{i=1}^{n-1} l_i[M_i] \quad \text{(ii)}$$
$$P_0 \mid \prod_{i=1}^{n} l_i[M_i] \xrightarrow{\alpha \in \{e, \overline{e}\}} P_0 \mid \prod_{i=1}^{n-1} l_i[M_i] \mid l_n[M'_n] \quad \text{(iii)}$$
$$\equiv P_0 \mid a[e \mid \overline{e}] \mid \prod_{i=1}^{n-1} l_i[M_i] \mid l_n[M'_n]$$
$$= P_0 \mid \prod_{i=1}^{n+1} l_i[M'_i]$$
$$\text{for } \ M'_i = M_i \ (0 \leq i \leq n-1),\ M_n \xrightarrow{\alpha} M'_n,\ l_{n+1} = a,\ M_{n+1} = e \mid \overline{e}.$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \xrightarrow{\gamma} Q_0 \mid a[0] \mid \prod_{i=1}^{n} l_i[N_i] \quad \text{(i)}$$
$$= Q_0 \mid \prod_{i=1}^{n+1} l_i[N_i] \quad \text{for } l_{n+1} = a,\ N_{n+1} = 0$$
$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \xrightarrow{\overline{l_n}\langle N_n \rangle} Q_0 \mid \prod_{i=1}^{n-1} l_i[N_i] \quad \text{(ii)}$$
$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \xrightarrow{\alpha} Q_0 \mid a[0] \mid \prod_{i=1}^{n} l_i[N_i] \quad \text{(iii)}$$
$$= Q_0 \mid \prod_{i=1}^{n+1} l_i[N_i] \quad \text{for } l_{n+1} = a,\ N_{n+1} = 0$$

Fig. 1. Simulation of observable transitions

cannot analyse the pairs: he can only use them along the tested processes P and Q which, by the design of environmental bisimulations, will make up for the differences.

Let us now observe the possible transitions from P and their corresponding transitions from Q by glossing over two pairs of trees, where related branches represent the correspondences. (Simulation in the other direction is similar and omitted for brevity.) First, let us consider the input and output actions as shown in Figure 1. (i) When P_0 does an input action e or an output action $\overline{e}$, it leaves behind a process $a[\overline{e}]$ or $a[e]$, respectively. Q_0 can also do the same action, leaving $a[0]$. Since both $(\overline{e}, 0)$ and $(e, 0)$ are in $\mathcal{E}$, we can add the leftover processes to the respective products $\prod$; (ii) output by passivation is trivial to match (without loss of generality, we only show the case $i = n$), and (iii) observable actions α of an M_n, leaving a residue M'_n, are matched by one of Q_0's $a[\alpha]$, leaving $a[0]$. To pair with this $a[0]$, we replicate an $a[e \mid \overline{e}]$ from P_0, and then, as in (i), they add up to the products $\prod$.

In a similar way, we explain how τ transitions of P are matched by Q, with another pair of transitions trees described in Figure 2.

(1) When an $a[e \mid \overline{e}]$ from P_0 turns into $a[0]$, Q does not have to do any action, for we work with weak bisimulations. By replication, Q can produce a copy $a[e]$ (or alternatively $a[\overline{e}]$) from Q_0, and since $(0, e)$ is in $\mathcal{E}$, we can add the $a[0]$ and the copy $a[e]$ to the products $\prod$; (2) P can also make a reaction between two copies of $a[e \mid \overline{e}]$ in P_0, leaving behind $a[e]$ and $a[\overline{e}]$. As in (1), Q can draw two copies of $a[e]$ from Q_0, and each product can be enlarged by two elements; (3) it is also possible for $M_n = e \mid \overline{e}$ to do a τ transition, becoming $M'_n = 0$. It stands that $(M'_n, N_n) \in \mathcal{E}$ and we are done; (4) very similarly, two processes M_n and M_{n-1} may react, becoming M'_n and M'_{n-1}. It stands also that (M'_{n-1}, N_{n-1}) and (M'_n, N_n) are in $\mathcal{E}$, so the resulting processes are still related; (5) it is possible for M_n to follow the transition $M_n \xrightarrow{\alpha} M'_n$ and react with

$$P_0 \mid \prod_{i=1}^{n} l_i[M_i] \longrightarrow$$

$$P_0 \mid a[0] \mid \prod_{i=1}^{n} l_i[M_i] \tag{1}$$

$$= P_0 \mid \prod_{i=1}^{n+1} l_i[M_i] \text{ for } l_{n+1} = a,\ M_{n+1} = 0$$

$$P_0 \mid a[e] \mid a[\overline{e}] \mid \prod_{i=1}^{n} l_i[M_i] \tag{2}$$

$$= P_0 \mid \prod_{i=1}^{n+2} l_i[M_i] \text{ for } l_{n+1} = l_{n+2} = a,\ M_{n+1} = e,\ M_{n+2} = \overline{e}$$

$$P_0 \mid \prod_{i=1}^{n} l_i[M_i'] \tag{3}$$

$$\text{for } M_i = M_i' \ (0 \leq i \leq n-1),\ M_n \xrightarrow{\tau} M_n'$$

$$P_0 \mid \prod_{i=1}^{n} l_i[M_i'] \tag{4}$$

$$\text{for } M_i = M_i' \ (0 \leq i \leq n-2),\ M_{n-1} \xrightarrow{e} M_{n-1}',\ M_n \xrightarrow{\overline{e}} M_n'$$

$$P_0 \mid a[\alpha] \mid \prod_{i=1}^{n} l_i[M_i'] \ = \ P_0 \mid \prod_{i=1}^{n+1} l_i[M_i'] \tag{5}$$

$$\text{for } M_i' = M_i \ (0 \leq i \leq n-1),\ M_n \xrightarrow{\alpha} M_n',\ \alpha \in \{e, \overline{e}\},\ l_{n+1} = a,\ M_{n+1} = \alpha$$

$$P_0 \mid a[\overline{e}] \mid \prod_{i=1}^{n-1} l_i[M_i] \ = \ P_0 \mid \prod_{i=1}^{n} l_i'[M_i'] \tag{6}$$

$$\text{for } l_i' = l_i,\ M_i' = M_i \ (0 \leq i \leq n-1),\ l_n' = a,\ M_n' = \overline{e}$$

$$P_0 \mid \prod_{i=1}^{n-2} l_i[M_i] \mid l_{n-1}[M_{n-1}'] \tag{7}$$

$$\equiv P_0 \mid a[e \mid \overline{e}] \mid \prod_{i=1}^{n-2} l_i[M_i] \mid l_{n-1}[M_{n-1}'] \ = \ P_0 \mid \prod_{i=1}^{n} l_i'[M_i']$$

$$\text{for } M_i' = M_i \ (0 \leq i \leq n-2),\ M_{n-1} \xrightarrow{e} M_{n-1}',\ l_i' = l_i \ (0 \leq i \leq n-1),\ l_n' = a,\ M_n' = e \mid \overline{e}$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \dashrightarrow$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \ \equiv \ Q_0 \mid a[e] \mid \prod_{i=1}^{n} l_i[N_i] \tag{1}$$

$$= Q_0 \mid \prod_{i=1}^{n+1} l_i[N_i] \text{ for } l_{n+1} = a, N_{n+1} = e$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \ \equiv \ Q_0 \mid a[e] \mid a[e] \mid \prod_{i=1}^{n} l_i[N_i] \tag{2}$$

$$= Q_0 \mid \prod_{i=1}^{n+2} l_i[N_i] \text{ for } l_{n+1} = l_{n+2} = a,\ N_{n+1} = N_{n+2} = e$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \tag{3}$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \tag{4}$$

$$Q_0 \mid \prod_{i=1}^{n} l_i[N_i] \ \equiv \ Q_0 \mid a[e] \mid \prod_{i=1}^{n} l_i[N_i] \tag{5}$$

$$= Q_0 \mid \prod_{i=1}^{n+1} l_i[N_i] \text{ for } l_{n+1} = a,\ N_{n+1} = e$$

$$Q_0 \mid a[0] \mid \prod_{i=1}^{n-1} l_i[N_i] \ = \ Q_0 \mid \prod_{i=1}^{n} l_i'[N_i'] \tag{6}$$

$$\text{for } l_i' = l_i,\ N_i' = N_i \ (0 \leq i \leq n-1),\ l_n' = a,\ N_n' = 0$$

$$Q_0 \mid a[0] \mid \prod_{i=1}^{n-1} l_i[N_i] \ = \ Q_0 \mid \prod_{i=1}^{n} l_i'[N_i'] \tag{7}$$

$$\text{for } l_i' = l_i,\ N_i' = N_i \ (0 \leq i \leq n-1),\ l_n' = a,\ N_n' = 0$$

Fig. 2. Simulation of internal transitions (dotted lines mean zero transitions)

a copy from P_0 which leaves behind $a[\alpha]$ (since $\overline{\alpha}$ has been consumed to conclude the reaction). Again, it stands that M'_n and N_n are related by $\mathcal{E}$, and that we can draw an $a[e]$ from Q_0 to pair it with the residue M'_n in the products $\prod$; (6) also, a copy $a[e \mid \overline{e}]$ from P_0 may passivate an $l_i[M_i]$, provided $l_i = e$, and leave a residue $a[\overline{e}]$. Q can do the same passivation using Q_0's $a[e]$, and leave $a[0]$. As it happens that $(\overline{e}, 0)$ is in $\mathcal{E}$, the residues can be added to the products too; (7) finally, the process $l_n[M_n]$, if $l_n = e$, may be passivated by M_{n-1}, reducing the size of P's product. Q can passivate $l_n[N_n]$ too, using a copy $a[e]$ from P_0, which becomes $a[0]$ after the reaction. Q's product too is shorter, but we need to add the $a[0]$ to it. To do so, we draw a copy $a[e \mid \overline{e}]$ from P_0, and since $(e \mid \overline{e}, 0)$ is in $\mathcal{E}$, $a[e \mid \overline{e}]$ and $a[0]$ are merged into their respective product.

This ends the sketch of the proof that $\mathcal{X}$ is an environmental bisimulation, and therefore that $!a[e \mid \overline{e}]$ and $!a[e] \mid a[\overline{e}]$ are behaviourally equivalent.

1.3 Overview of the paper

The rest of this paper is structured as follows. In Section 2 we describe the higher-order π-calculus with passivation. In Section 3 we formalize our environmental bisimulations. In Section 4 we give some examples of bisimilar processes. In Section 5, we bring up some future work to conclude our paper.

2 Higher-order π-calculus with passivation

We introduce a slight variation of the higher-order π-calculus with passivation [7]—HOπP for short—through its syntax and a labelled transitions system.

2.1 Syntax

The syntax of our HOπP processes P, Q is given by the following grammar, very similar to that of Lenglet *et al.* [7] (the higher-order π-calculus extended with located processes and their passivation):

$$\begin{array}{l} P, Q \ ::= 0 \mid a(X).P \mid \overline{a}\langle M\rangle.P \mid (P \mid P) \mid a[P] \mid \nu a.P \mid\ !P \mid \mathit{run}(M) \\ M, N ::= X \mid `P \end{array}$$

X ranges over the set of variables, and a over the set of names which can be used for both locations and channels. $a[P]$ denotes the process P running in location a. To define a general up-to context technique (Definition 2, see also Section 5), we distinguish terms M, N from processes P, Q and adopt explicit syntax for processes as terms $`P$ and their execution $\mathit{run}(M)$.

2.2 Labelled transitions system

We define n, *fn*, *bn* and *fv* to be the functions that return respectively the set of names, free names, bound names and free variables of a process or an action. We abbreviate a (possibly empty) sequence $x_0, x_1, \ldots, x_n$ as $\widetilde{x}$ for any meta-variable x. The transition semantics of HOπP is given by the following labelled transition system, which is based on that of the higher-order π-calculus (omitting symmetric rules PAR-R and REACT-R):

$$\frac{}{a(X).P \xrightarrow{a(M)} P\{M/X\}}\ \textsc{Ho-in} \qquad \frac{}{\overline{a}\langle M\rangle.P \xrightarrow{\overline{a}\langle M\rangle} P}\ \textsc{Ho-out}$$

$$\frac{P_1 \xrightarrow{\alpha} P_1' \quad bn(\alpha)\cap fn(P_2)=\emptyset}{P_1 \mid P_2 \xrightarrow{\alpha} P_1' \mid P_2}\ \textsc{Par-l} \qquad \frac{!P \mid P \xrightarrow{\alpha} P'}{!P \xrightarrow{\alpha} P'}\ \textsc{Rep}$$

$$\frac{P_1 \xrightarrow{(\nu\widetilde{b}).\overline{a}\langle M\rangle} P_1' \quad P_2 \xrightarrow{a(M)} P_2' \quad \{\widetilde{b}\}\cap fn(P_2)=\emptyset}{P_1 \mid P_2 \xrightarrow{\tau} \nu\widetilde{b}.(P_1' \mid P_2')}\ \textsc{React-l}$$

$$\frac{P \xrightarrow{\alpha} P' \quad a\notin n(\alpha)}{\nu a.P \xrightarrow{\alpha} \nu a.P'}\ \textsc{Guard} \qquad \frac{P \xrightarrow{(\nu\widetilde{b}).\overline{a}\langle M\rangle} P' \quad c\neq a \quad c\in fn(M)\setminus\{\widetilde{b}\}}{\nu c.P \xrightarrow{\nu(\widetilde{b},c).\overline{a}\langle M\rangle} P'}\ \textsc{Extr}$$

extended with the following three rules:

$$\frac{P \xrightarrow{\alpha} P'}{a[P] \xrightarrow{\alpha} a[P']}\ \textsc{Transp} \qquad \frac{}{a[P] \xrightarrow{\overline{a}\langle `P\rangle} 0}\ \textsc{Passiv} \qquad \frac{}{\mathit{run}(`P) \xrightarrow{\tau} P}\ \textsc{Run}$$

Assuming again knowledge of the standard higher-order π-calculus [9, 11], we only explain below the three added rules that are not part of it. The `Transp` rule expresses the *transparency* of locations, the fact that transitions can happen below a location and be observed outside its boundary. The `Passiv` rule illustrates that, at any time, a process running under a location can be passivated (stopped and turned into a term) and sent along the channel corresponding to the location's name. Quotation of the process output reminds us that higher-order communications transport terms. Finally, the `Run` rule shows how, at the cost of an internal transition, a process term be instantiated. As usual with small-steps semantics, transition does not progress for undefined cases (such as $\mathit{run}(X)$) or when the assumptions are not satisfied.

Henceforth, we shall write $\overline{a}.P$ to mean $\overline{a}\langle `0\rangle.P$ and $a.P$ for $a(X).P$ if $X\notin fv(P)$. We shall also write $\equiv$ for the structural congruence, whose definition is standard (see the appendix, Definition A.1).

3 Environmental bisimulations of HOπP

Given the higher-order nature of the language, and in order to get round the universal quantification issue of context bisimulations, we would like observations (terms) to be stored and reusable for further testing. To this end, let us define an *environmental relation* $\mathcal{X}$ as a set of elements $(r,\mathcal{E},P,Q)$ where r is a finite set of names, $\mathcal{E}$ is a binary relation (with finitely many free names) on variable-closed terms (i.e. terms with no free variables), and P and Q are variable-closed processes.

We generally write $x\oplus S$ to express the set union $\{x\}\cup S$. We also use graphically convenient notation $P\ \mathcal{X}_{\mathcal{E};r}\ Q$ to mean $(r,\mathcal{E},P,Q)\in\mathcal{X}$ and define the *term context closure* $(\mathcal{E};r)^\star=\mathcal{E}\cup\{(`P,`Q)\mid(P,Q)\in(\mathcal{E};r)^\circ\}$ with the *process context closure* $(\mathcal{E};r)^\circ=\{(C[\widetilde{M}],C[\widetilde{N}])\mid \widetilde{M}\mathcal{E}\widetilde{N},\ C\ \textit{context},\ bn(C)\cap fn(\mathcal{E},r)=\emptyset,\ fn(C)\subseteq r\}$, where a context is a process with zero or more *holes* for terms. Note the distinction of terms ‘P, ‘Q from processes P, Q. We point out that $(\emptyset; r)^\star$ is the identity on terms with free names in r, that $(\mathcal{E}; r)^\star$ includes $\mathcal{E}$ by definition, and that the context closure operations are monotonic on $\mathcal{E}$ (and r). Therefore, for any $\mathcal{E}$ and r, the set $(\mathcal{E}; r)^\star$ includes the identity $(\emptyset; r)^\star$ too. Also, we use the notations $\mathcal{S}.1$ and $\mathcal{S}.2$ to denote the first and second projections of a relation (i.e. set of pairs) S. Finally, we define weak transitions $\Rightarrow$ as the reflexive, transitive closure of $\xrightarrow{\tau}$, and $\xRightarrow{\alpha}$ as $\Rightarrow \xrightarrow{\alpha} \Rightarrow$ for $\alpha \neq \tau$ (and define $\xRightarrow{\tau}$ as $\Rightarrow$).

We can now define environmental bisimulations formally:

Definition 1. *An environmental relation $\mathcal{X}$ is an environmental bisimulation if $P \ \mathcal{X}_{\mathcal{E};r}\ Q$ implies:*

1. *if $P \xrightarrow{\tau} P'$, then $\exists Q'.\ Q \Rightarrow Q'$ and $P' \ \mathcal{X}_{\mathcal{E};r}\ Q'$,*
2. *if $P \xrightarrow{a(M)} P'$ with $a \in r$, and if $(M, N) \in (\mathcal{E}; r)^\star$, then $\exists Q'.\ Q \xRightarrow{a(N)} Q'$ and $P' \ \mathcal{X}_{\mathcal{E};r}\ Q'$,*
3. *if $P \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$ with $a \in r$ and $\widetilde{b} \notin fn(r, \mathcal{E}.1)$, then $\exists Q', N.\ Q \xRightarrow{\nu\widetilde{c}.\overline{a}\langle N\rangle} Q'$ with $\widetilde{c} \notin fn(r, \mathcal{E}.2)$ and $P' \ \mathcal{X}_{(M,N)\oplus\mathcal{E};r}\ Q'$,*
4. *for any $(‘P_1, ‘Q_1) \in \mathcal{E}$ and $a \in r$, we have $P \mid a[P_1] \ \mathcal{X}_{\mathcal{E};r}\ Q \mid a[Q_1]$,*
5. *for any $n \notin fn(\mathcal{E}, P, Q)$, we have $P \ \mathcal{X}_{\mathcal{E};n\oplus r}\ Q$, and*
6. *the converse of 1, 2 and 3 on Q's transitions.*

Modulo the symmetry resulting from clause 6, clause 1 is usual; clause 2 enforces bisimilarity to be preserved by any input that can be built from the knowledge, hence the use of the context closure; clause 3 enlarges the knowledge of the observer with the leaked out terms. Clause 4 allows the observer to spawn (and immediately run) terms concurrently to the tested processes, while clause 5 shows that he can also create fresh names at will.

A few points related to the handling of free names are worth mentioning: as the set of free names in $\mathcal{E}$ is finite, clause 5 can always be applied; therefore, the attacker can add arbitrary fresh names to the set r of known names so as to use them in terms M and N in clause 2. Fresh $\widetilde{b}$ and $\widetilde{c}$ in clause 3 also exist thanks to the finiteness of free names in $\mathcal{E}$ and r.

We define environmental bisimilarity $\sim$ as the union of all environmental bisimulations, and it holds that it is itself an environmental bisimulation (all the conditions above are monotone on $\mathcal{X}$). Therefore, $P \sim_{\mathcal{E};r} Q$ if and only if $P \ \mathcal{X}_{\mathcal{E};r}\ Q$ for some environmental bisimulation $\mathcal{X}$. We do particularly care about the situation where $\mathcal{E} = \emptyset$ and $r = fn(P, Q)$. It corresponds to the equivalence of two processes when the observer knows all of their free names (and thus can do all observations), but has not yet learnt any output pair.

For improving the practicality of our bisimulation proof method, let us devise an up-to context technique [11, p. 86]: for an environmental relation $\mathcal{X}$, we write $P \ \mathcal{X}^\star_{\mathcal{E};r}\ Q$ if $P \equiv \nu\widetilde{c}.(P_0 \mid P_1)$, $Q \equiv \nu\widetilde{d}.(Q_0 \mid Q_1)$, $P_0 \ \mathcal{X}_{\mathcal{E}';r'}\ Q_0$, $(P_1, Q_1) \in (\mathcal{E}'; r')^\circ$, $\mathcal{E} \subseteq (\mathcal{E}'; r')^\star$, $r \subseteq r'$, and $\{\widetilde{c}\} \cap fn(r, \mathcal{E}.1) = \{\widetilde{d}\} \cap fn(r, \mathcal{E}.2) = \emptyset$. As a matter of fact, this is actually an up-to context and up-to environment and up-to restriction and up-to structural congruence technique, but because of the clumsiness of this appellation we will restrain ourselves to "up-to context" to preserve clarity. To roughly explain the convenience behind this notation and its (long) name: (1) "up-to context" states that we can take any (P_1, Q_1) from the (process) context closure $(\mathcal{E}'; r')^\circ$ of the environment $\mathcal{E}'$ (with free names in r') and execute them in parallel with processes P_0 and Q_0 related by $\mathcal{X}_{\mathcal{E}';r'}$; similarly, we allow environments $\mathcal{E}$ with terms that are not in $\mathcal{E}'$ itself but are in the (term) context closure $(\mathcal{E}'; r')^\star$; (2) "up-to environment" states that, when proving the bisimulation clauses, we please ourselves with environments $\mathcal{E}'$ that are *larger* than the $\mathcal{E}$ requested by Definition 1; (3) "up-to restriction" states that we also content ourselves with tested processes P, Q with extra restrictions $\nu\widetilde{c}$ and $\nu\widetilde{d}$ (i.e. less observable names); (4) finally, "up-to structural congruence" states that we identify all processes that are structurally congruent to $\nu\widetilde{c}.(P_0 \mid P_1)$ and $\nu\widetilde{d}.(Q_0 \mid Q_1)$.

Using this notation, we define environmental bisimulations up-to context as follows:

Definition 2. *An environmental relation $\mathcal{X}$ is an environmental bisimulation up-to context if $P\ \mathcal{X}_{\mathcal{E};r}\ Q$ implies:*

1. *if $P \xrightarrow{\tau} P'$, then $\exists Q'.\ Q \Rightarrow Q'$ and $P'\ \mathcal{X}^\star_{\mathcal{E};r}\ Q'$,*
2. *if $P \xrightarrow{a(M)} P'$ with $a \in r$, and if $(M, N) \in (\mathcal{E}; r)^\star$, then $\exists Q'.\ Q \xRightarrow{a(N)} Q'$ and $P'\ \mathcal{X}^\star_{\mathcal{E};r}\ Q'$,*
3. *if $P \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$ with $a \in r$ and $\widetilde{b} \notin fn(r, \mathcal{E}.1)$, then $\exists Q', N.\ Q \xRightarrow{\nu\widetilde{c}.\overline{a}\langle N\rangle} Q'$ with $\widetilde{c} \notin fn(r, \mathcal{E}.2)$ and $P'\ \mathcal{X}^\star_{(M,N)\oplus\mathcal{E};r}\ Q'$,*
4. *for any $(`P_1, `Q_1) \in \mathcal{E}$ and $a \in r$, we have $P \mid a[P_1]\ \mathcal{X}^\star_{\mathcal{E};r}\ Q \mid a[Q_1]$,*
5. *for any $n \notin fn(\mathcal{E}, P, Q)$, we have $P\ \mathcal{X}_{\mathcal{E};n\oplus r}\ Q$, and*
6. *the converse of 1, 2 and 3 on Q's transitions.*

The conditions on each clause (except 5, which is unchanged for the sake of technical convenience) are weaker than that of the standard environmental bisimulations, as we require in the positive instances bisimilarity modulo a context, not just bisimilarity itself. It is important to remark that, unlike in [12] but as in [13], we do not need a specific context to avoid stating a tautology in clause 4; indeed, we spawn terms $(`P_1, `Q_1) \in \mathcal{E}$ immediately as processes P_1 and Q_1, while the context closure can only use the terms under an explicit *run* operator.

We prove the soundness (under some condition; see Remark 1) of environmental bisimulations as follows. Full proofs are found in the appendix, Section B but are nonetheless sketched below.

Lemma 1 (Input lemma). *If $(P_1, Q_1) \in (\mathcal{E}; r)^\circ$ and $P_1 \xrightarrow{a(M)} P_1'$ then $\forall N.\exists Q_1'.\ Q_1 \xrightarrow{a(N)} Q_1'$ and $(P_1', Q_1') \in ((M, N)\oplus\mathcal{E}; r)^\circ$.*

Lemma 2 (Output lemma). *If $(P_1, Q_1) \in (\mathcal{E}; r)^\circ$, $\{\widetilde{b}\} \cap fn(\mathcal{E}, r) = \emptyset$ and $P_1 \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P_1'$ then $\exists Q_1', N.\ Q_1 \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} Q_1'$, $(P_1', Q_1') \in (\mathcal{E}; \widetilde{b}\oplus r)^\circ$ and $(M, N) \in (\mathcal{E}; \widetilde{b}\oplus r)^\star$.*

Definition 3 (Run-erasure). *We write $P \leq Q$ if P can be obtained by (possibly repeatedly) replacing zero or more subprocesses run$(`R)$ of Q with R, and write $P\ \mathcal{Y}^-_{\mathcal{E};r}\ Q$ for $P \leq \mathcal{Y}^\star_{\leq\mathcal{E}\geq;r} \geq Q$.*

Definition 4 (Simple environment). *A process is called* simple *if none of its subprocesses has the form* $\nu a.P$ *or* $a(X).P$ *with* $X \in fv(P)$. *An environment is called simple if all the processes in it are simple. An environmental relation is called simple if all of its environments are simple (note that the tested processes may still be non-simple).*

Lemma 3 (Reaction lemma). *For any simple environmental bisimulation up-to context* $\mathcal{Y}$, *if* $P\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q$ *and* $P \xrightarrow{\tau} P'$, *then there is a* Q' *such that* $Q \stackrel{\tau}{\Rightarrow} Q'$ *and* $P'\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q'$.

Proof sketch. Lemma 1 (resp. 2) is proven by straightforward induction on the transition derivation of $P_1 \xrightarrow{a(M)} P_1'$ (resp. $P_1 \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P_1'$). Lemma 3 is proven last, as it uses the other two lemmas (for the internal communication case).

Lemma 4 (Soundness of up-to context). *Simple bisimilarity up-to context is included in bisimilarity.*

Proof sketch. By checking that $\{(r, \mathcal{E}, P, Q) \mid P\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q\}$ is included in $\sim$, where $\mathcal{Y}$ is the simple environmental bisimilarity up-to context. In particular, we use Lemma 1 for clause 2, Lemma 2 for clause 3, and Lemma 3 for clause 1 of the environmental bisimulation.

Our definitions of reduction-closed barbed equivalence $\approx$ and congruence $\approx_c$ are standard and omitted for brevity; see the appendix, Definition B.2 and B.3

Theorem 1 (Barbed equivalence from environmental bisimulation).
If $P\ \mathcal{Y}^{-}_{\emptyset;fn(P,Q)}\ Q$ *for a simple environmental bisimulation up-to context* $\mathcal{Y}$, *then* $P \approx Q$.

Proof sketch. By verifying that each clause of the definition of $\approx$ is implied by membership of $\mathcal{Y}^{-}$, using Lemma 4 for the parallel composition clause.

Corollary 1 (Barbed congruence from environmental bisimulation).
If $\overline{a}\langle`P\rangle\ \mathcal{Y}^{-}_{\emptyset;a\oplus fn(P,Q)}\ \overline{a}\langle`Q\rangle$ *for a simple environmental bisimulation up-to context* $\mathcal{Y}$, *then* $P \approx_c Q$.

We recall that, in context bisimulations, showing the equivalence of $\overline{a}\langle`P\rangle$ and $\overline{a}\langle`Q\rangle$ almost amounts to testing the equivalence of P and Q in every context. However, with environmental bisimulations, only the location context in clause 4 of the bisimulation has to be considered.

Remark 1. The extra condition "simple" is needed because of a technical difficulty in the proof of Lemma 3: when an input process $a(X).P$ is spawned under location b in parallel with an output context $\nu c.\overline{a}\langle M\rangle.Q$ (with $c \in fn(M)$), they can make the transition $b[a(X).P \mid \nu c.\overline{a}\langle M\rangle.Q] \xrightarrow{\tau} b[\nu c.(P\{M/X\} \mid Q)]$, where the restriction operator νc appears *inside* the location b (and therefore can be passivated together with the processes); however, our spawning clause only gives us $b[a(X).P] \mid \nu c.\overline{a}\langle M\rangle.Q \xrightarrow{\tau} \nu c.(b[P\{M/X\}] \mid Q)$ and does not cover the above case. Further investigation is required to overcome this difficulty (although we have not yet found a concrete counterexample of soundness, we conjecture some modification to the bisimulation clauses would be necessary). Note that, even if the environments are simple, the tested processes do not always have to be simple, as in Example 4 and 5. Moreover, thanks to up-to context, even the output terms (including passivated processes) can be non-simple.

4 Examples

Here, we give some examples of HOπP processes whose behavioural equivalence is proven with the help of our environmental bisimulations. In each example, we prove the equivalence by exhibiting a relation $\mathcal{X}$ containing the two processes we consider, and by showing that it is indeed a bisimulation up-to context (and environment, restriction and structural congruence). We write $P \mid \ldots \mid P$ for a finite, possibly null, product of the process P.

Example 1. $e \mid !a[e] \mid !a[0] \approx !a[e] \mid !a[0]$. (This example comes from [7].)

Proof. Take $\mathcal{X} = \{(r,\ \emptyset,\ e \mid P,\ P) \mid r \supseteq \{a, e\}\} \cup \{(r, \emptyset, P, P) \mid r \supseteq \{a, e\}\}$ where $P = !a[e] \mid !a[0]$. It is immediate to verify that whenever $P \xrightarrow{\alpha} P'$, we have $P' \equiv P$, and therefore that transition $e \mid P \xrightarrow{\alpha} e \mid P' \equiv e \mid P$ can be matched by $P \xrightarrow{\alpha} P' \equiv P$ and conversely. Also, for $e \mid P \xrightarrow{e} P$, we have that $P \xrightarrow{e} !a[e] \mid a[0] \mid !a[0] \equiv P$ and we are done since $(r, \emptyset, P, P) \in \mathcal{X}$. Moreover, the set r must contain the free names of P, and to satisfy clause 5 about adding fresh names, bigger r's must be allowed too. The passivations of $a[e]$ and $a[0]$ can be matched by syntactically equal actions with the pairs of output terms $(`e, `e)$ and $(`0, `0)$ included in the identity, which in turn is included in the context closure $(\emptyset; r)^{\star}$. Finally clause 4 of the bisimulation is vacuously satisfied because the environment is empty. We therefore have $e \mid !a[e] \mid !a[0] \approx !a[e] \mid !a[0]$ from the soundness of environmental bisimulation up-to context.

Example 2. $!\overline{a} \mid !e \approx !a[e]$.

Proof sketch. Take $\mathcal{X} = \{(r,\ \mathcal{E},\ P,\ Q) \mid r \supseteq \{a, e, l_1, \ldots, l_n\} \mid \mathcal{E} = \{(`0, `e)\},\ n \geq 0,$ $P = !\overline{a} \mid !e \mid \prod_{i=1}^{n} l_i[0],\ \ Q = !a[e] \mid \prod_{i=1}^{n} l_i[e] \mid a[0] \mid \ldots \mid a[0]\}$. See the appendix, Example C.1 for the rest of the proof.

Example 3. $!a[e] \mid !b[\overline{e}] \approx !a[b[e \mid \overline{e}]]$. This example shows the equivalence proof of more complicated processes with nested locations.

Proof sketch. Take:

$$\begin{aligned}
\mathcal{X} = \{(r,\ \mathcal{E},\ P,\ Q) \mid r\ \ &\supseteq \{a, e, b, l_1, \ldots, l_n\},\\
&P_0 = !a[e] \mid !b[\overline{e}],\ \ Q_0 = !a[b[e \mid \overline{e}]],\\
&P\ \ = P_0 \mid \textstyle\prod_{i=1}^{n} l_i[P_i] \mid b[0] \mid \ldots \mid b[0],\\
&Q\ \ = Q_0 \mid \textstyle\prod_{i=1}^{n} l_i[Q_i],\\
&(`\widetilde{P}, `\widetilde{Q}) \in \mathcal{E},\ n \geq 0\},\\
\mathcal{E}\ = \{(`x, `y) \mid x \in \{0, e, \overline{e}\},\ &y \equiv\in \{0, e, \overline{e}, (e \mid \overline{e}), b[0], b[e], b[\overline{e}], b[e \mid \overline{e}]\}\}\,.
\end{aligned}$$

See the appendix, Example C.2 for the rest of the proof.

Example 4. $c(X).\mathit{run}(X) \approx \nu f.(f[c(X).\mathit{run}(X)] \mid !f(Y).f[\mathit{run}(Y)])$. The latter process models a system where a process $c(X).\mathit{run}(X)$ runs in location f, and executes any process P it has received. In parallel is a process $f(Y).f[\mathit{run}(Y)]$ which can passivate $f[P]$ and respawn the process P under the same location f. Informally, this models a system which can restart a computer and resume its computation after a failure.

Proof. Take $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ where:

$$\begin{aligned}
\mathcal{X}_1 &= \{(r,\ \emptyset,\ c(X).\mathit{run}(X),\ \nu f.(f[c(X).\mathit{run}(X)] \mid !f(Y).f[\mathit{run}(Y)])) \mid r \supseteq \{c\}\},\\
\mathcal{X}_2 &= \{(r,\ \emptyset,\ P,\ Q) \mid r\ \supseteq c \oplus \mathit{fn}(R), \qquad S = \mathit{run}(`\mathit{run}(\ldots `\mathit{run}(`R)\ldots)),\\
&\qquad\qquad P \in \{\mathit{run}(`R), R\},\ \ Q = \nu \widetilde{f}.(f[S] \mid !f(Y).[\mathit{run}(Y)])\}.
\end{aligned}$$

As usual, we require that r contains at least the free name c of the tested processes. All outputs belong to $(\emptyset; r)^\star$ since they come from a process R drawn from $(\emptyset; r)^\star$, and therefore, we content ourselves with an empty environment $\emptyset$. Also, by the emptiness of the environment, clause 4 of environmental bisimulations is vacuously satisfied.

Verification of transitions of elements of $\mathcal{X}_1$, i.e. inputs of some $`R$ (with $(`R, `R) \in (\emptyset; r)^\star$) from $c$, is immediate and leads to checking elements of $\mathcal{X}_2$. For elements of $\mathcal{X}_2$, we observe that $P = \mathit{run}(`R)$ can do one τ transition to become R, while Q can do an internal transition passivating the process $\mathit{run}(`R)$ running in $f$ and place it inside $f[\mathit{run}(`\ )]$, again and again. Q can also do τ transitions that consume all the $\mathit{run}(`\ )$'s until it becomes $R$. Whenever $P$ (resp. $Q$) makes an observable transition, $Q$ (resp. $P$) can consume the $\mathit{run}(`\ )$'s and weakly do the same action as they exhibit the same process. We observe that all transitions preserve membership in $\mathcal{X}_2$ (thus in $\mathcal{X}$), and therefore we have that $\mathcal{X}$ is an environmental bisimulation up-to context, which proves the behavioural equivalence of the original processes $c(X).\mathit{run}(X)$ and $c(X).\nu f.(f[c(X).\mathit{run}(X)] \mid !f(Y).f[\mathit{run}(Y)])$.

Example 5. $c(X).\mathit{run}(X) \approx c(X).\nu a.(\overline{a}\langle X\rangle \mid !\nu f.(f[a(X).\mathit{run}(X)] \mid f(Y).\overline{a}\langle Y\rangle)))$. This example is a variation of Example 4 modelling a system where computation is resumed on another computer after a failure.

Proof. Take $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3$ where:

$$\begin{aligned}
\mathcal{X}_1 &= \{(r,\ \emptyset,\ c(X).\mathit{run}(X),\ c(X).\nu a.(\overline{a}\langle X\rangle \mid F)) \mid r \supseteq \{c\}\},\\
\mathcal{X}_2 &= \{(r,\ \emptyset,\ P_1,\ \nu a.(F \mid R_1 \mid R_2 \mid \overline{a}\langle `P_2\rangle)) \mid\\
&\quad r \supseteq \{c\} \oplus \mathit{fn}(P),\ \ P_1, P_2 \in \{\mathit{run}(`P), P\},\ \ R_1 = \overline{a}\langle N_1\rangle \mid \ldots \mid \overline{a}\langle N_n\rangle,\\
&\quad R_2 = \nu l_1.(l_1[Q_1] \mid l_1(Y).\overline{a}\langle Y\rangle) \mid \ldots \mid \nu l_m.(l_m[Q_m] \mid l_m(Y).\overline{a}\langle Y\rangle),\\
&\quad N_1, \ldots, N_n, `Q_1, \ldots, `Q_m = `\mathit{run}(`\mathit{run}(\ldots `\mathit{run}(`a(X).\mathit{run}(X))\ldots)),\ n \geq 0\},\\
\mathcal{X}_3 &= \{(r,\ \emptyset,\ P_1,\ \nu a.(F \mid R_1 \mid R_2 \mid \nu l.(l[P_2] \mid l(Y).\overline{a}\langle Y\rangle)))) \mid\\
&\quad r \supseteq \{c\} \oplus \mathit{fn}(P),\ \ P_1, P_2 \in \{\mathit{run}(`P), P\},\ \ R_1 = \overline{a}\langle N_1\rangle \mid \ldots \mid \overline{a}\langle N_n\rangle,\\
&\quad R_2 = \nu l_1.(l_1[Q_1] \mid l_1(Y).\overline{a}\langle Y\rangle) \mid \ldots \mid \nu l_m.(l_m[Q_m] \mid l_m(Y).\overline{a}\langle Y\rangle),\\
&\quad N_1, \ldots, N_n, `Q_1, \ldots, `Q_m = `\mathit{run}(`\mathit{run}(\ldots `\mathit{run}(`a(X).\mathit{run}(X))\ldots)),\ n \geq 0\},\\
F\ &= !\nu f.(f[a(X).\mathit{run}(X)] \mid f(Y).\overline{a}\langle Y\rangle).
\end{aligned}$$

The set of names r and the environment share the same fate as those of Example 4 for identical reasons. For ease, we write *lhs* and *rhs* to conveniently denote each of the tested processes.

Verification of the bisimulation clauses of $\mathcal{X}_1$ is immediate and leads to a member $(r, \emptyset, \mathit{run}(`P), \nu a.(\overline{a}\langle `P\rangle \mid F))$ of $\mathcal{X}_2$ for some $`P$ with $(`P, `P) \in (\emptyset; r)^\star$. For $\mathcal{X}_2$, *lhs* can do an internal action (consuming its outer $\mathit{run}(`\ )$) that *rhs* does not have to follow since we work with weak bisimulations, and the results is still in $\mathcal{X}_2$; conversely, internal actions of *rhs* do not have to be matched. Some of those transitions that *rhs* can do are reactions between replications from F. All those transitions creates elements of either R_1 or R_2 that can do nothing but internal actions and can be ignored further in the proof thanks to the weakness of our bisimulations.

Whenever *lhs* does an observable action α, that is, when $P_1 = P \xrightarrow{\alpha} P'$, *rhs* must do a reaction between $\overline{a}\langle`P_2\rangle$ and F, giving $\nu l.(l[P_2] \mid l(Y).\overline{a}\langle Y\rangle) \overset{\alpha}{\Rightarrow} \nu l.(l[P'] \mid l(Y).\overline{a}\langle Y\rangle)$ which satisfies $\mathcal{X}_3$'s definition. Moreover, all transitions of P_1 or P_2 in $\mathcal{X}_3$ can be matched by the other, hence preserving the membership in $\mathcal{X}_3$. Finally, a subprocess $\nu l.(l[P_2] \mid l(Y).\overline{a}\langle Y\rangle)$ of *rhs* of $\mathcal{X}_3$ can do a τ transition to $\overline{a}\langle`P_2\rangle$ and the residues belong back to $\mathcal{X}_2$.

This concludes the proof of behavioural equivalence of the original processes $c(X).\mathit{run}(X)$ and $c(X).\nu a.(\overline{a}\langle X\rangle.!\nu f.(f[a(X).\mathit{run}(X)] \mid f(Y).f[\mathit{run}(Y)]))$.

5 Discussion and future work

In the original higher-order π-calculus with passivation described by Lenglet *et al.* [7], terms are identified with processes: its syntax is just $P ::= 0 \mid X \mid a(X).P \mid \overline{a}\langle P\rangle.P \mid (P \mid P) \mid a[P] \mid \nu a.P \mid !P$. We conjecture that it is also possible to develop sound environmental bisimulations (and up-to context, etc.) for this version of HOπP, as we [12] did for the standard higher-order π-calculus. However we chose not to cover directly the original higher-order π-calculus with passivation, for two reasons: (1) the proof method of [12] which relies on guarded processes and a factorisation trick using the spawning clause of the bisimulation is inadequate in the presence of locations; (2) there is a very strong constraint in clause 4 of up-to context in [12, Definition E.1 (Appendix)] (the context has no hole for terms from $\mathcal{E}$). By distinguishing processes from terms, not only is our up-to context method much more general, but our proofs are also direct and technically simple. Although one might argue that the presence of the *run* operator is a burden, by using Definition 3, one could devise an "up-to *run*" technique and abstract $\mathit{run}(\ldots `\mathit{run}(`P))$ as P, making equivalence proofs easier to write and understand.

As described in Remark 1, removing the limitation on the environments is left for future work. We also plan to apply environmental bisimulations to (a substantial subset of) the Kell calculus so that we can provide a practical alternative to context bisimulations in a more expressive higher-order distributed process calculus. In the Kell calculus, locations are not transparent: one discriminates messages on the grounds of their origins (i.e. from a location above, below, or from the same level). For example, consider the (simplified) Kell processes $P = \overline{a}\langle M\rangle.!b[\overline{a}]$ and $Q = \overline{a}\langle N\rangle.!b[\overline{a}]$ where $M = \overline{a}$ and $N = 0$. They seem bisimilar assuming environmental bisimulations naively like those in this paper: intuitively, both P and Q can output (respectively M and N) to channel a, and their continuations are identical; passivation of spawned $l[M]$ and $l[N]$ for known location l would be immediately matched; finally, the output to channel a under l, turning P's spawned $l[M]$ into $l[0]$, could be matched by an output to a under b by Q's replicated $b[\overline{a}]$. However, M and N behave differently when observed from the same level (or below), for example as in $l[M \mid a(Y).\overline{ok}]$ and $l[N \mid a(Y).\overline{ok}]$ even under the presence of $!b[\overline{a}]$. More concretely, the context $[\cdot]_1 \mid a(X).c[X \mid a(Y).\overline{ok}]$ distinguishes P and Q, showing the unsoundness of such naive definition. This suggests that, to define sound environmental bisimulations in Kell-like calculi with non-transparent locations, we should require a stronger condition such as bisimilarity of M and N in the output clause. Developments on this idea are in progress.

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Appendix for “Sound Bisimulations for Higher-Order Distributed Process Calculus”

Adrien Piérard and Eijiro Sumii

Tohoku University
{adrien,sumii}@kb.ecei.tohoku.ac.jp

A Higher-order π-calculus with passivation

1 Syntax

The syntax of HOπP processes P, Q is given by the following grammar:

$$\begin{array}{ll} P, Q & ::= 0 \ \mid\ a(X).P \ \mid\ \overline{a}\langle M\rangle.P \ \mid\ (P \mid P) \ \mid\ a[P] \ \mid\ \nu a.P \ \mid\ !P \ \mid\ \mathit{run}(M) \\ M, N, V, W & ::= X \ \mid\ `P \end{array}$$

We define the functions that returns the free names and free variables respectively as:

$$\begin{array}{llll}
fn(0) & = \emptyset & fv(0) & = \emptyset \\
fn(a(X).P) & = \{a\} \cup fn(P) & fv(a(X).P) & = fv(P) \setminus \{X\} \\
fn(\overline{a}\langle M\rangle.P) & = \{a\} \cup fn(M) \cup fn(P) & fv(\overline{a}\langle M\rangle.P) & = fv(M) \cup fv(P) \\
fn(P_1 \mid P_2) & = fn(P_1) \cup fn(P_2) & fv(P_1 \mid P_2) & = fv(P_1) \cup fv(P_2) \\
fn(a[P]) & = \{a\} \cup fn(P) & fv(a[P]) & = fv(P) \\
fn(\nu a.P) & = fn(P) \setminus \{a\} & fv(\nu a.P) & = fv(P) \\
fn(!P) & = fn(P) & fv(!P) & = fv(P) \\
fn(\mathit{run}(M)) & = fn(M) & fv(\mathit{run}(M)) & = fv(M) \\
fn(X) & = \emptyset & fv(X) & = \{X\} \\
fn(`P) & = fn(P) & fv(`P) & = fv(P)
\end{array}$$

We conveniently write $fn(X, Y, \ldots, Z)$ (resp. $fv(X, Y, \ldots, Z)$) to denote $\bigcup_{S\in\{X,Y,\ldots,Z\}} fn(S)$ (resp. $\bigcup_{S\in\{X,Y,\ldots,Z\}} fv(S)$).

2 Labelled transitions system

The transitions semantics of HOπP is given by the following labelled transitions system:

$$\frac{}{a(X).P \xrightarrow{a(M)} P\{M/X\}}\ \textsc{Ho-in} \qquad \frac{}{\overline{a}\langle M\rangle.P \xrightarrow{\overline{a}\langle M\rangle} P}\ \textsc{Ho-out}$$

$$\frac{P_1 \xrightarrow{\alpha} P_1' \quad bn(\alpha) \cap fn(P_2) = \emptyset}{P_1 \mid P_2 \xrightarrow{\alpha} P_1' \mid P_2}\ \textsc{Par-l} \qquad \frac{P_2 \xrightarrow{\alpha} P_2' \quad bn(\alpha) \cap fn(P_1) = \emptyset}{P_1 \mid P_2 \xrightarrow{\alpha} P_1 \mid P_2'}\ \textsc{Par-r}$$

$$\frac{P_1 \xrightarrow{(\nu\widetilde{b}).\overline{a}\langle M\rangle} P_1' \quad P_2 \xrightarrow{a(M)} P_2' \quad \{\widetilde{b}\} \cap fn(P_2) = \emptyset}{P_1 \mid P_2 \xrightarrow{\tau} \nu\widetilde{b}.(P_1' \mid P_2')}\ \textsc{React-l}$$

$$\frac{P_1 \xrightarrow{a(M)} P_1' \quad P_2 \xrightarrow{(\nu\widetilde{b}).\overline{a}\langle M\rangle} P_2' \quad \{\widetilde{b}\} \cap fn(P_1) = \emptyset}{P_1 \mid P_2 \xrightarrow{\tau} \nu\widetilde{b}.(P_1' \mid P_2')} \text{ REACT-R}$$

$$\frac{P \xrightarrow{\alpha} P' \quad a \notin n(\alpha)}{\nu a.P \xrightarrow{\alpha} \nu a.P'} \text{ GUARD} \qquad \frac{!P \mid P \xrightarrow{\alpha} P'}{!P \xrightarrow{\alpha} P'} \text{ REP}$$

$$\frac{P \xrightarrow{(\nu\widetilde{b}).\overline{a}\langle M\rangle} P' \quad c \neq a \quad c \in fn(M) \setminus \{\widetilde{b}\}}{\nu c.P \xrightarrow{\nu(\widetilde{b},c).\overline{a}\langle M\rangle} P'} \text{ EXTR}$$

$$\frac{P \xrightarrow{\alpha} P'}{a[P] \xrightarrow{\alpha} a[P']} \text{ TRANSP} \qquad \frac{}{a[P] \xrightarrow{\overline{a}\langle `P\rangle} 0} \text{ PASSIV} \qquad \frac{}{run(`P) \xrightarrow{\tau} P} \text{ RUN}$$

with the following functions on labels

$$n(\alpha) = \begin{cases} \emptyset & \text{if } \alpha = \tau \\ \{a\} \cup fn(V) & \text{if } \alpha = a(V) \\ \{a, \widetilde{b}\} \cup fn(V) & \text{if } \alpha = \nu\widetilde{b}.\overline{a}\langle V\rangle \end{cases} \qquad bn(\alpha) = \begin{cases} \emptyset & \text{if } \alpha = \tau \text{ or } \alpha = a(V) \\ \{\widetilde{b}\} & \text{if } \alpha = \nu\widetilde{b}.\overline{a}\langle V\rangle \end{cases}$$

and the notation $\widetilde{x}$ to denote the sequence $x_1, x_2, \ldots x_n$.

Definition A.1. *Structural congruence $\equiv$ is the smallest relation on processes such that:*

$$\frac{Q \equiv P}{P \equiv Q} \text{ S-SYM} \qquad \frac{}{P \equiv P} \text{ S-REFL} \qquad \frac{P \equiv R \quad R \equiv Q}{P \equiv Q} \text{ S-TRANS}$$

$$\frac{}{P \equiv P \mid 0} \text{ S-EMPTY} \qquad \frac{}{P_1 \mid (P_2 \mid P_3) \equiv (P_1 \mid P_2) \mid P_3} \text{ S-ASSOC} \qquad \frac{}{P_1 \mid P_2 \equiv P_2 \mid P_1} \text{ S-COMMUT}$$

$$\frac{}{\nu a.0 \equiv 0} \text{ S-NULL} \qquad \frac{}{\nu a.\nu b.P \equiv \nu b.\nu a.P} \text{ S-SWAP} \qquad \frac{a \notin fn(P_1)}{P_1 \mid (\nu a.P_2) \equiv \nu a.(P_1 \mid P_2)} \text{ S-SCOPE}$$

$$\frac{P \equiv Q}{\nu a.P \equiv \nu a.Q} \text{ S-GUARD} \qquad \frac{P \equiv Q}{a(X).P \equiv a(X).Q} \text{ S-IN} \qquad \frac{P_1 \equiv Q_1 \quad P_2 \equiv Q_2}{\overline{a}\langle `P_1\rangle.P_2 \equiv \overline{a}\langle `Q_1\rangle.Q_2} \text{ S-OUT}$$

$$\frac{}{!P \equiv !P \mid P} \text{ S-REP} \qquad \frac{P \equiv Q}{!P \equiv !Q} \text{ S-BANG} \qquad \frac{P_1 \equiv Q_1 \quad P_2 \equiv Q_2}{P_1 \mid P_2 \equiv Q_1 \mid Q_2} \text{ S-COMP}$$

$$\frac{P \equiv Q}{a[P] \equiv a[Q]} \text{ S-LOC} \qquad \frac{P \equiv Q}{run(`P) \equiv run(`Q)} \text{ S-RUN}$$

Definition A.2. *Structural congruence on labels $\equiv$ is defined by:*

$$\frac{}{\tau \equiv \tau} \text{ L-TAU} \qquad \frac{M \equiv N}{a(M) \equiv a(N)} \text{ L-IN} \qquad \frac{M \equiv N}{\nu\widetilde{c}.\overline{a}\langle M\rangle \equiv \nu\widetilde{c}.\overline{a}\langle N\rangle} \text{ L-OUT}$$

Lemma A.3. [Reduction preserves structural congruence]
If $P \equiv Q$ then

(a) *for all α, P', if $P \xrightarrow{\alpha} P'$ then either*

  i. *there are $\widetilde{c}$, a, M such that if $\alpha \equiv \nu\widetilde{c}.\overline{a}\langle M\rangle$ or $\alpha \equiv \tau$, then there are β, Q' such that $Q \xrightarrow{\beta} Q'$, $\alpha \equiv \beta$ and $P' \equiv Q'$, or*

  ii. *there are a, M such that if $\alpha \equiv a(M)$, then for all β such that $\alpha \equiv \beta$, there is Q' such that $Q \xrightarrow{\beta} Q'$ and $P' \equiv Q'$, and*

(b) *for all α, Q', if $Q \xrightarrow{\alpha} Q'$ then either*

  i. *there are $\widetilde{c}$, a, M such that if $\alpha \equiv \nu\widetilde{c}.\overline{a}\langle M\rangle$ or $\alpha \equiv \tau$, then there are β, P' such that $P \xrightarrow{\beta} P'$, $\alpha \equiv \beta$ and $P' \equiv Q'$, or*

  ii. *there are a, M such that if $\alpha \equiv a(M)$, then for all β such that $\alpha \equiv \beta$, there is P' such that $P \xrightarrow{\beta} P'$ and $P' \equiv Q'$.*

Proof. By induction on the derivations of $P \equiv Q$.

B Environmental bisimulations of HOπP

1 Notations

Definition B.1. [Contexts]
We define contexts for terms C (contexts that have holes for terms) and contexts for processes C_p (contexts that have holes for processes) as

$$\begin{array}{lcl} D_p & ::= & X \mid `C_p \\ C_p & ::= & [\cdot]_i \mid 0 \mid a(X).C_p \mid \overline{a}\langle D_p\rangle.C_p \mid (C_p \mid C_p) \mid a[C_p] \mid \nu a.C_p \mid !C_p \mid \mathit{run}(D_p) \end{array}$$

$$\begin{array}{lcl} D & ::= & [\cdot]_i \mid X \mid `C \\ C & ::= & 0 \mid a(X).C \mid \overline{a}\langle D\rangle.C \mid (C \mid C) \mid a[C] \mid \nu a.C \mid !C \mid \mathit{run}(D) \end{array}$$

Unless explicitly specified otherwise, the word "context" will denote a context for terms.

Definition B.2. [Reduction-closed barbed equivalence]
Reduction-closed barbed equivalence $\approx$ is the largest binary relation on variable-closed processes such that when $P \approx Q$,

- *$P \xrightarrow{\tau} P'$ implies $\exists Q'.\ Q \Rightarrow Q'$ and $P' \approx Q'$,*
- *$P \downarrow_\mu$ implies $Q \Downarrow_\mu$,*
- *the converse of the above two on Q, and*
- *$\forall R.\ P \mid R \approx Q \mid R$.*

Definition B.3. [Reduction-closed barbed congruence]
Reduction-closed barbed congruence $\approx_c$ is the largest binary relation on variable-closed processes such that when $P \approx_c Q$,

- *$P \xrightarrow{\tau} P'$ implies $\exists Q'.\ Q \Rightarrow Q'$ and $P' \approx_c Q'$,*
- *$P \downarrow_\mu$ implies $Q \Downarrow_\mu$,*
- *the converse of the above two on Q, and*
- *for all C context with holes for processes, $C[P] \approx_c C[Q]$.*

2 Soundness of environmental bisimulations

Lemma B.4. [Originally Lemma 1 "Input lemma"]
If $(P_1, Q_1) \in (\mathcal{E}; r)^\circ$ *and* $P_1 \xrightarrow{a(M)} P_1'$ *then* $\forall N. \exists Q_1'.\ Q_1 \xrightarrow{a(N)} Q_1' \wedge (P_1', Q_1') \in ((M,N) \oplus \mathcal{E}; r)^\circ$.

Proof. By induction on the transition derivation $P_1 \xrightarrow{a(M)} P_1'$. There are six cases to check.

1. **Case** IN: $C = a(X).C_1$
   We have that $P_1 = a(X).C_1[\widetilde{M}] \xrightarrow{a(M)} C_1[\widetilde{M}]\{M/X\}$ and that $Q_1 = a(X).C_1[\widetilde{N}] \xrightarrow{a(N)} C_1[\widetilde{N}]\{N/X\}$. We are done since we replace term X by terms M and N, hence $C_1[\widetilde{M}]\{M/X\}((M,N) \oplus \mathcal{E}; r)^\circ C_1[\widetilde{N}]\{N/X\}$.
2. **Case** PAR-L: $C = C_1 \mid C_2$
   We have that $P_1 = C_1[\widetilde{M}] \mid C_2[\widetilde{M}] \xrightarrow{a(M)} P' \mid C_2[\widetilde{M}]$, i.e. $C_1[\widetilde{M}] \xrightarrow{a(M)} P'$. By the induction hypothesis $C_1[\widetilde{N}] \xrightarrow{a(N)} Q'$ and $P'((M,N) \oplus \mathcal{E}; r)^\circ Q'$, from which we derive $(P' \mid C_2[\widetilde{M}])((M,N) \oplus \mathcal{E}; r)^\circ (Q' \mid C_2[\widetilde{N}])$ as well as $C_1[\widetilde{N}] \mid C_2[\widetilde{N}] \xrightarrow{a(N)} Q' \mid C_2[\widetilde{N}]$.
3. **Case** PAR-R: $C = C_1 \mid C_2$
   Similar.
4. **Case** TRANSP: $C = l[C_1]$
   We have that $P_1 = l[C_1[\widetilde{M}]] \xrightarrow{a(M)} l[P']$, that is $C_1[\widetilde{M}] \xrightarrow{a(M)} P'$. By the induction hypothesis, we have that $C_1[\widetilde{N}] \xrightarrow{a(N)} Q'$ and $P'((M,N) \oplus \mathcal{E}; r)^\circ Q'$, from which we derive $l[P']((M,N) \oplus \mathcal{E}; r)^\circ l[Q']$ as well as $l[C_1[\widetilde{N}]] \xrightarrow{a(N)} l[Q']$.
5. **Case** GUARD: $C = \nu b.C_1$
   We have that $P_1 = \nu b.C_1[\widetilde{M}] \xrightarrow{a(M)} \nu b.P'$, i.e. $C_1[\widetilde{M}] \xrightarrow{a(M)} P'$, $b \notin n(a, M)$ and $C_1[\widetilde{M}](\mathcal{E}; b \oplus r)^\circ C_1[\widetilde{N}]$. By the induction hypothesis, we have $C_1[\widetilde{N}] \xrightarrow{a(N)} Q'$ and $P'((M,N) \oplus \mathcal{E}; b \oplus r)^\circ Q'$, hence $\nu b.P'((M,N) \oplus \mathcal{E}; r)^\circ \nu b.Q'$. Finally, $\nu b.C_1[\widetilde{N}] \xrightarrow{a(N)} Q'$.
6. **Case** REP: $C = !C_1$
   We have that $P_1 = !C_1[\widetilde{M}] \xrightarrow{a(M)} P'$, i.e. $!C_1[\widetilde{M}] | C_1[\widetilde{M}] \xrightarrow{a(M)} P'$. By the induction hypothesis, we have that $!C_1[\widetilde{N}] \mid C_1[\widetilde{N}] \xrightarrow{a(N)} Q'$ and $P'((M,N) \oplus \mathcal{E}; r)^\circ Q'$. Thus $!C_1[\widetilde{N}] \xrightarrow{a(N)} Q'$ and still $P'((M,N) \oplus \mathcal{E}; r)^\circ Q'$.

Lemma B.5. [Originally Lemma 2, "Output lemma"]
If $P_1\ (\mathcal{E}; r)^\circ\ Q_1$, $\{\widetilde{b}\} \cap fn(\mathcal{E}, r) = \emptyset$ *and* $P_1 \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P_1'$ *then* $\exists Q_1', N.\ Q_1 \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} Q_1'$, $P_1'(\mathcal{E}; \widetilde{b} \oplus r)^\circ Q_1'$ *and* $M\ (\mathcal{E}; \widetilde{b} \oplus r)^\star\ N$.

Proof. By induction on the transition derivation $P_1 \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P_1'$. There are eight cases to check.

1. **Case** OUTPUT: $C = \overline{a}\langle `C_1\rangle.C_2$

   We have that $P_1 = \overline{a}\langle `C_1[\widetilde{M}]\rangle.C_2[\widetilde{M}] \xrightarrow{\overline{a}\langle `C_1[\widetilde{M}]\rangle} C_2[\widetilde{M}]$ and that $Q_1 = \overline{a}\langle `C_1[\widetilde{N}]\rangle.C_2[\widetilde{N}] \xrightarrow{\overline{a}\langle `C_1[\widetilde{N}]\rangle} C_2[\widetilde{N}]$. It is immediate to confirm that $`C_1[\widetilde{M}](\mathcal{E};r)^\star$ $`C_1[\widetilde{N}]$ and $C_2[\widetilde{M}]\ (\mathcal{E};r)^\circ\ C_2[\widetilde{N}]$ hold.
2. **Case** PAR-L: $C = C_1 \mid C_2$

   We have that $P_1 = C_1[\widetilde{M}] \,|\, C_2[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P' \,|\, C_2[\widetilde{M}]$, i.e. $C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$ and $\{\widetilde{b}\} \cap \mathit{fn}(C_2[\widetilde{M}]) = \emptyset$. By the induction hypothesis, we have that $C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} Q'$ and $P'(\mathcal{E};(\widetilde{b}\oplus r))^\circ Q'$ and $M\ (\mathcal{E};(\widetilde{b}\oplus r))^\star\ N$. Since $\widetilde{b} \notin \mathit{fn}(C_2[\widetilde{M}])$–i.e. $\widetilde{b} \notin \mathit{fn}(C_2)$–and $\widetilde{b} \notin \mathcal{E}$, we have that $C_1[\widetilde{N}] \mid C_2[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} Q' \mid C_2[\widetilde{N}]$, and $(P' \mid C_2[\widetilde{M}])(\mathcal{E};(\widetilde{b}\oplus r))^\circ(Q' \mid C_2[\widetilde{N}])$.
3. **Case** PAR-R: $C = C_1 \mid C_2$
   Similar.
4. **Case** TRANSP: $C = l[C_1]$

   We have that $P_1 = l[C_1[\widetilde{M}]] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} l[P']$, i.e. $C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$. By the induction hypothesis, we have $C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.a\langle N\rangle} Q'$, $P'(\mathcal{E};(\widetilde{b}\oplus r))^\circ Q'$ and $M\ (\mathcal{E};(\widetilde{b}\oplus r))^\star$ N. From this we derive $l[C_1[\widetilde{N}]] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} l[Q']$ and $l[P'](\mathcal{E};(\widetilde{b}\oplus r))^\circ l[Q']$ and we are done.
5. **Case** PASSIV: $C = l[C_1]$

   We have that $P_1 = l[C_1[\widetilde{M}]] \xrightarrow{\overline{l}\langle `C_1[\widetilde{M}]\rangle} 0$. Immediately, we have $Q_1 = l[C_1[\widetilde{N}]] \xrightarrow{\overline{l}\langle `C_1[\widetilde{N}]\rangle}$ 0 with $`C_1[\widetilde{M}]\ (\mathcal{E};r)^\star\ `C_1[\widetilde{N}]$ and $0\ (\mathcal{E};r)^\circ\ 0$.
6. **Case** GUARD: $C = \nu c.C_1$
   By the definition of process context closure, it holds that $C_1[\widetilde{M}](\mathcal{E};c\oplus r)^\circ C_1[\widetilde{N}]$. We also have that $P_1 = \nu c.C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} \nu c.P'$, i.e. $C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$ and $c \notin \{\widetilde{b}, a\} \cup \mathit{fn}(M)$. By the induction hypothesis, we have $C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.a\langle N\rangle} Q'$, $C_0[\widetilde{M}] = M(\mathcal{E};(\widetilde{b}\oplus c\oplus r))^\star N = C_0[\widetilde{N}]$ and $P'(\mathcal{E};(\widetilde{b}\oplus c\oplus r))^\circ Q'$, hence $\nu c.P'(\mathcal{E};(\widetilde{b}\oplus r))^\circ \nu c.Q'$. Moreover, since $c \notin \mathit{fn}(M)$, we have $c \notin \mathit{fn}(C_0)$, and since $c \notin \mathit{fn}(\mathcal{E})$ it also holds that $c \notin \mathit{fn}(C_0[\widetilde{N}])$, hence $\nu c.C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} \nu c.Q'$ and $M\ (\mathcal{E};(\widetilde{b}\oplus r))^\star\ N$.
7. **Case** REP: $C = !C_1$

   We have that $P_1 = !C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$, i.e. $!C_1[\widetilde{M}] \mid C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$. By the induction hypothesis, we have $!C_1[\widetilde{N}] \mid C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.a\langle N\rangle} Q'$, $M\ (\mathcal{E};(\widetilde{b}\oplus r))^\star\ N$ and $P'(\mathcal{E};(\widetilde{b}\oplus r))^\circ Q'$, hence $!C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} Q'$ and we are done.
8. **Case** EXTR: $C = \nu c.C_1$

   We have that $P_1 = \nu c.C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b},c.\overline{a}\langle M\rangle} P'$, i.e. $C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$ with $C_1[\widetilde{M}](\mathcal{E};c\oplus r)^\circ C_1[\widetilde{N}]$ $a \neq c$, $c \in \mathit{fn}(M) \setminus \{\widetilde{b}\}$. By the induction hypothesis, we have $C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b}.\overline{a}\langle N\rangle} Q'$ and $M\ (\mathcal{E};(\widetilde{b}\oplus c\oplus r))^\star\ N$ and $P'(\mathcal{E};(\widetilde{b}\oplus c\oplus r))^\circ Q'$. Since $c \in \mathit{fn}(M) \setminus \{\widetilde{b}\}$, we have $c \in \mathit{fn}(C_0[\widetilde{M}]) \setminus \{\widetilde{b}\}$. Since $c \notin \mathit{fn}(r,\mathcal{E})$, by the definition of context closure necessarily $c \in fn(C_0) \setminus \{\widetilde{b}\}$, hence $c \in fn(C_0[\widetilde{N}]) \setminus \{\widetilde{b}\}$ i.e. $c \in fn(N) \setminus \{\widetilde{b}\}$. We are then done since $\nu c.C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{b},c.\overline{a}\langle N\rangle} Q'$.

Lemma B.6. [Input and output preserve environmental bisimulation up-to context] *Let $\mathcal{Y}$ be an environmental bisimulation up-to context and $\mathcal{X} = \{(r, \mathcal{E}, P, Q) \mid P\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q\}$. Then, for all $P\ \mathcal{X}_{\mathcal{E};r}\ Q$,*

1. *if $P \xrightarrow{a(V)} P'$ with a in r, then for all $(V, W) \in (\mathcal{E}; r)^{\star}$ there is a Q' such that $Q \xRightarrow{a(W)} Q'$ and $P'\ \mathcal{X}_{\mathcal{E};r}\ Q'$,*
2. *if $P \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle V\rangle} P'$ with a in r and $\widetilde{c}_1 \notin fn(\mathcal{E}.1, r)$, then there is a Q' such that $Q \xRightarrow{\nu\widetilde{d}_1.\overline{a}\langle W\rangle} Q'$ with $\widetilde{d}_1 \notin fn(\mathcal{E}.2, r)$ and $P'\ \mathcal{X}_{(V,W)\oplus\mathcal{E};r}\ Q'$, and*
3. *the converse of the above two hold for Q's transitions too.*

Proof. Suppose $P\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q$, therefore for some P_0, P_1, Q_0, Q_1, $\mathcal{E}'$, r', $\widetilde{c}$, $\widetilde{d}$, we have $P \equiv \nu\widetilde{c}.(P_0 \mid P_1)$, $Q \equiv \nu\widetilde{d}.(Q_0 \mid Q_1)$, $r \subseteq r'$, $\{\widetilde{c}\} \cap fn(r, \mathcal{E}.1) = \{\widetilde{d}\} \cap fn(r, \mathcal{E}.2) = \emptyset$, $\mathcal{E} \subseteq (\mathcal{E}'; r')^{\star}$, $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$ and $P_1\ (\mathcal{E}'; r')^{\circ}\ Q_1$.

We are going to analyse all the possible input/output transitions.

1. **Case**: Input
   There are two cases for this transition:
   (a) **Subcase**: $P_0 \xrightarrow{a(V)} P_0' \quad \{\widetilde{c}\} \cap n(a(V)) = \emptyset$
   By $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, we have $Q_0 \xRightarrow{a(W)} Q_0'$ and $P_0'\ \mathcal{Y}^{\star}_{\mathcal{E}';r'}\ Q_0'$. It then holds that $\nu\widetilde{c}.(P_0 \mid P_1) \xrightarrow{a(V)} \equiv \nu\widetilde{c}, \widetilde{c}_1.(P_{00}' \mid P_{01}' \mid P_1) \equiv P'$. Also, since $V\ (\mathcal{E}; r)^{\star}\ W$, we have that $fn(W) \subseteq fn(r, \mathcal{E}.2)$, and since $\widetilde{d} \cap fn(r, \mathcal{E}.2) = \emptyset$, we have that $\nu\widetilde{d}.(Q_0 \mid Q_1) \xRightarrow{a(W)} \equiv \nu\widetilde{d}, \widetilde{d}_1.(Q_{00}' \mid Q_{01}' \mid Q_1) \equiv Q'$. $P'\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q'$ follows, hence $P'\ \mathcal{X}_{\mathcal{E};r}\ Q'$.
   (b) **Subcase**: $P_1 \xrightarrow{a(V)} P_1' \quad \{\widetilde{c}\} \cap n(a(V)) = \emptyset$
   By Lemma B.4, we have that $Q_1 \xrightarrow{a(W)} Q_1'$ and $P_1'\ ((V, W) \oplus \mathcal{E}'; r')^{\circ}\ Q_1'$. Since $(\mathcal{E}; r)^{\star} \subseteq (\mathcal{E}'; r')^{\star}$, we actually have $P_1'\ (\mathcal{E}'; r')^{\circ}\ Q_1'$. Then, we have $\nu\widetilde{c}.(P_0 \mid P_1) \xrightarrow{a(V)} \nu\widetilde{c}.(P_0 \mid P_1') \equiv P'$ and $\nu\widetilde{d}.(Q_0 \mid Q_1) \xrightarrow{a(W)} \nu\widetilde{d}.(Q_0 \mid Q_1') \equiv Q'$ since $\widetilde{d} \notin fn(r, \mathcal{E}.2)$. Therefore, $P'\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q'$, that is, $P'\ \mathcal{X}_{\mathcal{E};r}\ Q'$.
2. **Case**: Output
   There are two cases for this transition:
   (a) **Subcase**: $P_0 \xrightarrow{\nu\widetilde{i}.\overline{a}\langle V\rangle} P_0'$
   We have $\{\widetilde{i}\} = \{\widetilde{c}_1\} \setminus \{\widetilde{c}\}$, that is $\{\widetilde{i}\} \notin fn(r, \mathcal{E}.1)$. In order to apply clause 3 of the bisimulation, we may need to substitute a fresh variable not in $fn(r', \mathcal{E}')$ for $\widetilde{i}$ in P_0' and V below, and we will assume that it has been done to release some burden from the proof. Let $P' = \nu\widetilde{c}_r.(P_0' \mid P_1)$ and $\widetilde{i}, \widetilde{c}_o = \widetilde{c}_1$. By $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, we have $Q_0 \xRightarrow{\nu\widetilde{j}.\overline{a}\langle W\rangle} Q_0'$, $\widetilde{j} \notin fn(r', \mathcal{E}'.2)$ and $P_0'\ \mathcal{Y}^{\star}_{(V,W)\oplus\mathcal{E}';r'}\ Q_0'$. Also, we have that $\nu\widetilde{d}.(Q_0 \mid Q_1) \xRightarrow{\nu\widetilde{j},\widetilde{d}_o.\overline{a}\langle W\rangle} \nu\widetilde{d}_r.(Q_0' \mid Q_1)$ with $\{\widetilde{d}_o\} \subseteq \{\widetilde{d}\}$ and $\widetilde{d}_o \in fn(W) \setminus \{\widetilde{j}\}$, and we then define $\widetilde{d}_1 = \widetilde{j}, \widetilde{d}_o$. By $P_0'\ \mathcal{Y}^{\star}_{(V,W)\oplus\mathcal{E}';r'}\ Q_0'$, we have

- $P'_0 = \nu\widetilde{c}_2.(P_{00} \mid P_{01})$, $Q'_0 = \nu\widetilde{d}_2.(Q_{00} \mid Q_{01})$,
- $\widetilde{c}_2 \notin fn(\mathcal{E}'.1, r')$, $\widetilde{d}_2 \notin fn(\mathcal{E}'.2, r')$,
- $P_{00}\ \mathcal{Y}_{\mathcal{E}'';r''}\ Q_{00}$, $P_{01}\ (\mathcal{E}''; r'')^\circ\ Q_{01}$,
- $(V, W)\oplus\mathcal{E}' \subseteq (\mathcal{E}''; r'')^\star$, $r' \subseteq r''$.

Since $P_1\ (\mathcal{E}'; r')^\circ\ Q_1$, we have $(P_1 \mid P_{01})\ (\mathcal{E}''; r'')^\circ\ (Q_1 \mid Q_{01})$. Since $\mathcal{E} \subseteq (\mathcal{E}'; r')^\star$, we have $(V, W)\oplus\mathcal{E} \subseteq ((V, W)\oplus\mathcal{E}'; r')^\star \subseteq (\mathcal{E}''; r'')^\star$. Finally, $\widetilde{c}_r, \widetilde{c}_2 \notin fn(\mathcal{E}.1, r)$ and $\widetilde{d}_r, \widetilde{d}_2 \notin fn(\mathcal{E}.2, r)$, $P' \equiv \nu\widetilde{c}_r, \widetilde{c}_2.(P_{00} \mid P_{01} \mid P_1)$ and $Q' \equiv \nu\widetilde{d}_r, \widetilde{d}_2.(Q_{00} \mid Q_{01} \mid Q_1)$, from where we conclude that $P\ \mathcal{Y}^\star_{(V,W)\oplus\mathcal{E};r}\ Q$, that is, after undoing the potential substitution, $P\ \mathcal{X}_{(V,W)\oplus\mathcal{E};r}\ Q$.

(b) **Subcase**: $P_1 \xrightarrow{\nu\widetilde{i}.\overline{a}\langle V\rangle} P'_1$

We have $\{\widetilde{i}\} \subseteq \{\widetilde{c}_1\}$, that is, $\widetilde{i} \notin fn(r, \mathcal{E}.1)$ and $\nu\widetilde{c}.(P_0 \mid P'_1) \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle V\rangle} \nu\widetilde{c}_r.(P_0 \mid P'_1) = P'$ with $\{\widetilde{c}_r\} = \{\widetilde{c}\} \setminus \{\widetilde{c}_1\}$. In order to apply Lemma B.5, we may need to substitute a fresh variable not in $fn(r', \mathcal{E}')$ for $\widetilde{i}$ in P'_1, Q'_1, V and W below, and we will assume that it has been done to release some burden from the proof.

By Lemma B.5, we have that $Q_1 \xrightarrow{\nu\widetilde{i}.\overline{a}\langle W\rangle} Q'_1$ and $V\ (\mathcal{E}'; \widetilde{i}\oplus r')^\star\ W$ as well as $P'_1\ (\mathcal{E}'; \widetilde{i}\oplus r')^\circ\ Q'_1$. Then, since we can assume $\{\widetilde{i}\} \cap fn(Q_0) = \emptyset$, we have $Q_0 \mid Q_1 \xrightarrow{\nu\widetilde{i}.\overline{a}\langle W\rangle} Q_0 \mid Q'_1$, and therefore $\nu\widetilde{d}.(Q_0 \mid Q_1) \xrightarrow{\nu\widetilde{i},\widetilde{d}_o} \nu\widetilde{d}_r.(Q_0 \mid Q'_1) \equiv Q'$, with $\widetilde{d}_1 = \widetilde{i}, \widetilde{d}_o$ and $\{\widetilde{d}_1\} \cap fn(r, \mathcal{E}.2) = \emptyset$. So far, we have

- $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, that is $P_0\ \mathcal{Y}_{\mathcal{E}';\widetilde{i}\oplus r'}\ Q_0$ since $\{\widetilde{i}\} \cap fn(\mathcal{E}', r') = \emptyset$,
- $P'_1\ (\mathcal{E}'; \widetilde{i}\oplus r')^\circ\ Q'_1$,
- $\mathcal{E} \subseteq (\mathcal{E}'; r')^\star \subseteq (\mathcal{E}'; \widetilde{i}\oplus r')^\star$,
- $r \subseteq r' \subseteq \widetilde{i}\oplus r'$.
- $\{\widetilde{c}_r\} \cap fn(r, \mathcal{E}.1) = \{\widetilde{d}_r\} \cap fn(r, \mathcal{E}.2) = \emptyset$

hence $P'\ \mathcal{Y}^\star_{\mathcal{E};r}\ Q'$ and therefore, after undoing the potential substitution, $P'\ \mathcal{X}_{\mathcal{E};r}\ Q'$.

3. **Case**: The converse of the above two cases on Q's transitions.
Similar to clauses 1 and 2.

Definition B.7. [*run*-erasure and *run*-expansion]
For all processes A, B, we write $A < B$ and $B > A$ if there are C_p and $\widetilde{R}$ such that $A = C_p[\widetilde{R}]$ and $B = C_p[run`(\widetilde{R})]$. We write $P_0 \leq P_n$ if $P_0 < \cdots < P_n$ for some $n \geq 0$. We naturally write $A \geq B$ whenever $B \leq A$. We naturally extend $\leq$ and $\geq$'s definitions to terms and labels.

We use the metavariables P^+ and P^- along with P when we mean that $P \leq P^+$ and that $P^- \leq P$. (The notations $(\cdot)^+$ and $(\cdot)^-$ therefore do not represent operators.) Similarly, we use the metavariables M^+ and M^- to represent run-expansions and run-erasures of term M.

Lemma B.8. [Input, output and reduction preserve $<$ and $>$]
Let $\mathcal{X} = \{(r, \mathcal{E}, P, Q) \mid P < Q,\ \mathcal{E} \subseteq\ <\}$. If $P\ \mathcal{X}_{\mathcal{E};r}\ Q$, then

- *if $P \xrightarrow{\tau} P'$ then there is a Q' such that $P'\ \mathcal{X}_{\mathcal{E};r}\ Q'$ and either $Q \xrightarrow{\tau} Q'$ or $Q \xrightarrow{run} \xrightarrow{\tau} Q'$,*
- *if $P \xrightarrow{a(M)} P'$ then for all $(M, N) \in (\mathcal{E}; r)^\star$ there is a Q' such that $P'\ \mathcal{X}_{\mathcal{E};r}\ Q'$ and either $Q \xrightarrow{a(N)} Q'$ or $Q \xrightarrow{run} \xrightarrow{a(N)} Q'$,*

– *if* $P \xrightarrow{\nu\widetilde{c}.\overline{a}\langle M\rangle} P'$ *then there are* Q', $M < N$ *such that* $P' \ \mathcal{X}_{(M,N)\oplus,\mathcal{E};r}\ Q'$ *and either* $Q \xrightarrow{\nu\widetilde{c}.\overline{a}\langle N\rangle} Q'$ *or* $Q \xrightarrow{run} \xrightarrow{\nu\widetilde{c}.\overline{a}\langle N\rangle} Q'$*, and*
– *the converse on* Q*'s transitions.*

Similarly for $>$ *.*

Proof. By induction on the derivation transition of $C_p[\widetilde{R}]$ (or $C_p[run(`\widetilde{R})]$). We only show the HO-IN derivation case of the input preservation, the others being straightforward or similar.

– **Case** P's input:
  There are two subcases: the context inputs, or some R_i.
  • $P = C_p[\widetilde{R}] \xrightarrow{a(M)} C'_p[\widetilde{R}, M]$ by an input from the context, and thus for $(M,N) = (C''_p[\widetilde{A}], C''_p[run`\widetilde{A}]) \in\ <$ , $Q = C_p[run`\widetilde{R}] \xrightarrow{a(N)} C'_p[run`\widetilde{R}, run`C''_p[run`\widetilde{A}]] \xrightarrow{\tau} C'_p[run`\widetilde{R}, C''_p[run`\widetilde{A}]] = C'''_p[run`\widetilde{R}, run`\widetilde{A}]$. We are done as $C'_p[\widetilde{R}, M] = C'_p[\widetilde{R}, C''_p[`\widetilde{A}]] = C''''_p[\widetilde{R}, \widetilde{A}] < C''''_p[run`\widetilde{R}, run`\widetilde{A}]$.
  • $P = C_p[\widetilde{R}, R] \xrightarrow{a(M)} C_p[\widetilde{R}, R']$ by an input from R, and thus for $(M,N) = (C''_p[\widetilde{A}], C''_p[run`\widetilde{A}]) \in\ <$ , we have $Q = C_p[run`\widetilde{R}, run`R] \xrightarrow{\tau} C_p[run`\widetilde{R}, R] \xrightarrow{a(N)} C_p[run`\widetilde{R}, R'']$. Since $(R', R'') = (C''''_p[\widetilde{A}], C''''_p[run`\widetilde{A}])$, we have $R' < R''$ and therefore $C_p[\widetilde{R}, R'] < C_p[run`\widetilde{R}, R'']$ and we are done.

– **Case** Q's input:
  There is only one subcase, as no R_i can input for they are all guarded.
  • $Q = C_p[run`\widetilde{R}] \xrightarrow{a(N)} C'_p[run`\widetilde{R}, N]$ by an input from the context, and thus for $(M,N) = (C''_p[\widetilde{A}], C''_p[run`\widetilde{A}]) \in\ <$ , $P = C_p[`\widetilde{R}] \xrightarrow{a(M)} C'_p[\widetilde{R}, C''_p[\widetilde{A}]] = C''''_p[\widetilde{R}, \widetilde{A}]$. We are done as $C'_p[run`\widetilde{R}, N] = C'_p[run`\widetilde{R}, C''_p[run`\widetilde{A}]] = C''''_p[run`\widetilde{R}, run`\widetilde{A}] > C''''_p[\widetilde{R}, \widetilde{A}]$.

Corollary B.9. [Input, output and reduction preserve $\leq$ and $\geq$]
For all r*, for any closed* P*,* Q*, the sets* $\mathcal{X}_1 = \{(r, \mathcal{E}, P_0, P_m) \mid \mathcal{E} \subseteq\ \leq,\ P_0 \leq P_m\}$*, and* $\mathcal{X}_2 = \{(r, \mathcal{E}, P_0, P_m) \mid \mathcal{E} \subseteq\ \geq,\ P_0 \geq P_m\}$ *are both preserved by input, output and reduction.*

Proof. By induction on the number n of $<$ (in $\leq$) (resp. on the number of $>$ (in $\geq$)). We explicitly treat only the set $\mathcal{X}_1$, the set $\mathcal{X}_2$ being treated similarly.

– **Case** $n = 0$
  Trivial.
– **Case** $n > 0$
  We have $P_0 \xrightarrow{\alpha} P'_0$, hence by Lemma B.8 either
  • $P_1 \xrightarrow{\beta} P'_1$ with $\alpha < \beta$ and $P'_0 < P'_1$. By $P_1 \leq P_n$ (which has $n-1$ "<"), applying the induction hypothesis we have $P_n \stackrel{\gamma}{\Rightarrow} P'_n$ with $\beta \leq \gamma$, hence $\alpha \leq \gamma$, and $P'_0 \leq P'_n$, or

- $P_1 \xrightarrow{\tau} P_1'' \xrightarrow{\beta} P_1'$ with $\alpha < \beta$ and $P_0' < P_1'$. By $P_1 \leq P_n$ (which has $n-1$ "<"), applying the induction hypothesis with $P_1 \xrightarrow{\tau} P_1''$, we have $P_n \stackrel{\tau}{\Rightarrow} P_n''$ and $P_1'' \leq P_n''$. Then we apply again the induction hypothesis to $P_1'' \leq P_n''$ and $P_1'' \xrightarrow{\beta} P_1'$, and we obtain $P_n'' \stackrel{\gamma}{\Rightarrow} P_n''$ and $P_1' \leq P_n'$ with $\beta \leq \gamma$, hence $P_0' \leq P_n'$ and $\alpha \leq \gamma$.

Definition B.10. [*run*-transition]
We write $P \xrightarrow{run} P'$ when $P \xrightarrow{\tau} P'$ is derived using the rule RUN. *Then, we write $P_0 \xRightarrow{run^n} P_n$ to mean that $P_0 \xrightarrow{run} \ldots \xrightarrow{run} P_n$.*

Definition B.11. [Minimal transition of *run*-expanded processes]
Suppose that $A \leq B$, $A \xrightarrow{\alpha} A'$, $B \xRightarrow{run^n} \xrightarrow{\beta} B'$ with $\alpha \leq \beta$ and that $A' \leq B'$. We say that $B \xRightarrow{run^n} \xrightarrow{\beta} B'$ is minimal with respect to $A \xrightarrow{\alpha} A'$ if and only if for all $B \xRightarrow{run^m} \xrightarrow{\gamma} B''$ with $A' \leq B''$ and $\alpha \leq \gamma$, we have $n \leq m$.

Lemma B.12. [Minimality and *run*-transition]
Suppose that $B \xrightarrow{run} B'' \xRightarrow{run^{n-1}} \xrightarrow{\beta} B'$ with $n > 0$ is minimal with respect to $A \xrightarrow{\alpha} A'$. We have that $B'' \xRightarrow{run^{n-1}} \xrightarrow{\beta} B'$ too is minimal with respect to $A \xrightarrow{\alpha} A'$.

Proof. By *reductio ad absurdum*. Suppose that $B'' \xRightarrow{run^{n-1}} \xrightarrow{\beta} B'$ is not minimal with respect to $A \xrightarrow{\alpha} A'$. There must be a minimal transition $B'' \xRightarrow{run^m} \xrightarrow{\gamma} B'''$ with $A' \leq B'''$, $\alpha \leq \gamma$, and $m < n-1$. Then we have a derivation $B \xrightarrow{run} B'' \xRightarrow{run^m} \xrightarrow{\gamma} B'''$ of length $m+1 < n$ with $A' \leq B'''$, which contradicts the assumption that $B \xrightarrow{run} B'' \xRightarrow{run^{n-1}} \xrightarrow{\beta} B'$ is minimal.

Lemma B.13. [Minimality and contexts]
For all $Q \xRightarrow{run^n} \xrightarrow{\beta} Q'$ minimal with respect to $P \xrightarrow{\alpha} P'$,

- *for all $R_1 \geq R_2$, $Q \mid R_1 \xRightarrow{run^n} \xrightarrow{\beta} Q' \mid R_1$ is minimal with respect to $P \mid R_2 \xrightarrow{\alpha} P' \mid R_2$,*
- *for l, $l[Q] \xRightarrow{run^n} \xrightarrow{\beta} l[Q']$ is minimal with respect to $l[P] \xrightarrow{\alpha} l[P']$, and*
- *if $Q = Q_0 \mid Q_1$, $Q' = Q_0' \mid Q_1'$, and $P = P_0 \mid P_1$ with $P_0 \leq Q_0$, $P_1 \leq Q_1$, then for all l and m, $l[Q_0] \mid m[Q_1] \xRightarrow{run^n} \xrightarrow{\beta} l[Q_0'] \mid m[Q_1']$ is minimal with respect to $l[P_0] \mid m[P_1] \xrightarrow{\alpha} l[P_0'] \mid m[P_1']$.*
- *for all $\widetilde{c}$, $\nu\widetilde{c}.Q \xRightarrow{run^n} \xrightarrow{\beta'} \nu\widetilde{c}'.Q''$ is minimal with respect to $P \xrightarrow{\alpha'} P''$.*

Proof. Immediate, as none of the above operations can reduce the number of *run*'s that have to be deleted, and as they all preserve membership to $\leq$.

Definition B.14. [*run*-erased context closure]
We define the run-erased context closure $(\mathcal{E}; r)^-$ of environment $\mathcal{E}$ with names r as $\leq (\mathcal{E}; r)^\star \geq$, that is $\{(M,N) \mid M \leq A,\ N \leq B,\ (A,B) \in (\mathcal{E}; r)^\star\}$. Notice that $(\mathcal{E}; r)^-$ may erase run's inside elements related by $\mathcal{E}$ too.

We also write $P\, \mathcal{Y}^-_{\mathcal{E};r}\, Q$ if $P \leq \mathcal{Y}^\star_{\leq\mathcal{E}\geq;r} \geq Q$ (which implies $\mathcal{Y}^\star \subseteq \mathcal{Y}^-$). In other words $P\, \mathcal{Y}^-_{\mathcal{E};r}\, Q$ if $P \equiv \nu\widetilde{c}.(P_0|P_1)$, $Q \equiv \nu\widetilde{d}.(Q_0|Q_1)$, $P_0 \leq \mathcal{Y}_{\mathcal{E}';r'} \geq Q_0$, $(`P_1, `Q_1) \in (\mathcal{E}; r)^-$, $\mathcal{E} \subseteq (\mathcal{E}'; r')^-$, $r \subseteq r'$, and $\{\widetilde{c}\} \cap fn(\mathcal{E}.1, r) = \{\widetilde{d}\} \cap fn(\mathcal{E}.2, r) = \emptyset$.

Corollary B.15. [*run*-erasure preserves *run*-erased context closure of environmental bisimulation up-to context]
If $P\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q$*,* $P^{-} \leq P$*,* $Q^{-} \leq Q$ *and* $\mathcal{E}^{-} \leq \mathcal{E}$ *then* $P^{-}\ \mathcal{Y}^{-}_{\mathcal{E}^{-};r}\ Q^{-}$*.*

Proof. From transitivity of $\leq$ and $\geq$ given by Definition B.7.

Lemma B.16. [Addition of fresh names preserves environmental bisimulation up-to context and its *run*-erased context closure]
Let $\mathcal{Y}$ *be an environmental bisimulation up-to context. If* $P\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q$ *and* $l \notin fn(P,Q,\mathcal{E})$*, then* $P\ \mathcal{Y}^{\star}_{\mathcal{E};l\oplus r}\ Q$*. Similarly, if* $P\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q$ *and* $l \notin fn(P,Q,\mathcal{E})$*, then* $P\ \mathcal{Y}^{-}_{\mathcal{E};l\oplus r}\ Q$*.*

Proof. By simple set arithmetic and use of definitions.

- **Case** $\mathcal{Y}^{\star}$
  Given $P = \nu\widetilde{c}.(P_0 \mid P_1)$, $Q = \nu\widetilde{d}.(Q_0 \mid Q_1)$ such that $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, $(P_1, Q_1) \in (\mathcal{E}'; r')^{\circ}$, $\{\widetilde{c}\} \cap fn(\mathcal{E}.1, r) = \{\widetilde{d}\} \cap fn(\mathcal{E}.2, r) = \emptyset$, and $\mathcal{E} \subseteq (\mathcal{E}'; r')^{\star}$, it holds that
  - $P_0\ \mathcal{Y}_{\mathcal{E}';l\oplus r'}\ Q_0$ by clause 5 of environmental bisimulation up-to context,
  - $(P_1, Q_1) \in (\mathcal{E}'; r')^{\circ} \subseteq (\mathcal{E}'; l\oplus r')^{\circ}$,
  - $\mathcal{E} \subseteq (\mathcal{E}'; r')^{\star} \subseteq (\mathcal{E}'; l\oplus r')^{\star}$
  - $l\oplus r \subseteq l\oplus r'$,
  - we can use renaming of $\widetilde{c}$ and $\widetilde{d}$ so that they do not clash with l.

  Therefore, $P\ \mathcal{Y}^{\star}_{\mathcal{E};l\oplus r}\ Q$ holds.
- **Case** $\mathcal{Y}^{-}$
  We have some $P^{+} \geq P$, $Q^{+} \geq Q$, $\mathcal{E}^{+} \geq \mathcal{E}$ such that $P^{+}\ \mathcal{Y}^{\star}_{\mathcal{E}^{+};r}\ Q^{+}$. Therefore, according to the above case, we have $P^{+}\ \mathcal{Y}^{\star}_{\mathcal{E}^{+};n\oplus r}\ Q^{+}$, hence $P\ \mathcal{Y}^{-}_{\mathcal{E};n\oplus r}\ Q$ by Definition B.14.

Lemma B.17. [Spawning preserves context closure of environmental bisimulation up-to context]
Let $\mathcal{Y}$ *be an environmental bisimulation up-to context. For all* $P\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q$*,* $l \in r$ *and* $(`P_2, `Q_2) \in \mathcal{E}$*, we have* $P \mid l[P_2]\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q \mid l[Q_2]$*.*

Proof. We have $P \equiv \nu\widetilde{c}.(P_0|P_1)$ and $Q \equiv \nu\widetilde{d}.(Q_0|Q_1)$, with $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, $(P_1, Q_1) \in (\mathcal{E}'; r')^{\circ}$, $\mathcal{E} \subseteq (\mathcal{E}'; r')^{\star}$, $r \subseteq r'$ and $\nu\widetilde{c} \notin fn(\mathcal{E}.1, r)$, $\nu\widetilde{d} \notin fn(\mathcal{E}.2, r)$. By $(`P_2, `Q_2) \in \mathcal{E}$, we have either $(`P_2, `Q_2) \in \mathcal{E}'$ or $(P_2, Q_2) \in (\mathcal{E}'; r')^{\circ}$. In the former case, it holds that $P_0 \mid l[P_2]\ \mathcal{Y}^{\star}_{\mathcal{E}';r'}\ Q_0 \mid l[Q_2]$ by clause 4 of environmental bisimulation up-to context, hence $\nu\widetilde{c}.(P_0 \mid l[P_2] \mid P_1)\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ \nu\widetilde{d}.(Q_0 \mid l[Q_2] \mid Q_1)$ up-to environment, context and restriction. In the latter case, we immediately have $(P_1 \mid l[P_2], Q_1 \mid l[Q_2]) \in (\mathcal{E}'; r')^{\circ}$, hence $P \mid l[P_2]\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q \mid l[Q_2]$.

Lemma B.18. [*run*-transitions of $(\mathcal{E}; r)^{\circ}$]
Suppose that $(P_1, Q_1) = (C[\widetilde{M}], C[\widetilde{N}]) \in (\mathcal{E}; r)^{\circ}$ *and that* $P_1 \xrightarrow{run} P_1'$*. Then there is a* Q_1' *such that* $Q_1 \xrightarrow{run} Q_1'$ *and either* $(P_1', Q_1') = (C'[\widetilde{M}], C'[\widetilde{N}]) \in (\mathcal{E}; r)^{\circ}$ *or* $(P_1', Q_1') = (C_p[run(\widetilde{M'}), A], C_p[run(\widetilde{N'}), B]) \in (\mathcal{E}; r)^{-} \backslash (\mathcal{E}; r)^{\circ}$ *with* (A, B) *in redex position (i.e. not under a run, an* $a(\cdot)$ *or an* $\overline{a}\langle\cdot\rangle$*) and* $(\widetilde{M'}, `A) = \widetilde{M}$*,* $(\widetilde{N'}, `B) = \widetilde{N}$*.*

Proof. By induction on the transition derivation of $P_1 \xrightarrow{run} P1'$. The only case of interest is the RUN one, developed below. The others (PAR-L, PAR-R, GUARD, REP and TRANSP) are straightforward.

1. **Case** RUN: $C = run(C_1)$
   There are two subcases
   (a) $C_1 = `C_2$
   We have $P_1 = run(`C_2[\widetilde{M}]) \xrightarrow{run} C_2[\widetilde{M}]$ and $Q_1 = run(`C_2[\widetilde{N}]) \xrightarrow{run} C_2[\widetilde{N}]$ with $(C_2[\widetilde{M}], C_2[\widetilde{N}]) \in (\mathcal{E}; r)^\circ$.
   (b) $C_1 = [\cdot]$
   We have $P_1 = run(`A) \xrightarrow{run} A$, $Q_1 \xrightarrow{run} B$ with $(`A, `B) \in \mathcal{E}$ (we can assume that $(A, B) \notin (\mathcal{E}; r)^\circ$, otherwise we could have handled this situation in the above subcase), and $(P_1', Q_1') = (C_p[A], C_p[B]) \in (\mathcal{E}; r)^- \setminus (\mathcal{E}; r)^\circ$ with $C_p = [\cdot]_1$, and (A, B) in redex position.

Lemma B.19. [Non-*run* τ-transitions of $(\mathcal{E}; r)^\circ$]
Suppose that $(P_1, Q_1) \in (\mathcal{E}; r)^\circ$ and that $P_1 \xrightarrow{\tau} P_1'$ is not a run-transition. Then there is a Q_1' such that $Q_1 \xrightarrow{\tau} Q_1'$ and $(P_1', Q_1') = (\mathcal{E}; r)^\circ$.

Proof. By induction on the transition derivation $P_1 \xrightarrow{\tau} P_1'$. The only interesting cases are REACT-R and REACT-L. The others (PAR-R, PAR-L, GUARD, TRANSP and REP are straightforward.)

1. **Case** REACT-L: $C = C_1 \mid C_2$
   We have $P_1 = C_1[\widetilde{M}] \mid C_2[\widetilde{M}] \xrightarrow{\tau} \nu\widetilde{c}.(P_{11} \mid P_{12})$ with $C_1[\widetilde{M}] \xrightarrow{\nu\widetilde{c}.\overline{a}\langle V\rangle} P_{11}$ and $C_2[\widetilde{M}] \xrightarrow{a(V)} P_{12}$, $\{\widetilde{c}\} \cap fn(C_2[\widetilde{M}]) = \emptyset$. We can assume $\{\widetilde{c}\} \cap fn(r, \mathcal{E}) = \emptyset$. By Lemma B.5, we know that $C_1[\widetilde{N}] \xrightarrow{\nu\widetilde{c}.\overline{\langle W\rangle}} Q_{11}$, that $P_{11}(\mathcal{E}; (\widetilde{c}\oplus r))^\circ Q_{11}$, and that $V\ (\mathcal{E}; (\widetilde{c}\oplus r))^\star\ W$. By Lemma B.4, we have that $C_2[\widetilde{N}] \xrightarrow{a(W)} Q_{12}$ with $P_{12}((V, W)\oplus\mathcal{E}; r)^\circ Q_{12}$. Also, since $\{\widetilde{c}\} \cap fn(C_2[\widetilde{N}]) = \emptyset$ (as $\widetilde{c} \notin fn(r, \mathcal{E})$), we have that $C_1[\widetilde{N}] \mid C_2[\widetilde{N}] \xrightarrow{\tau} \nu\widetilde{c}.(Q_{11} \mid Q_{12})$. Moreover, by $P_{11}(\mathcal{E}; (\widetilde{c}\oplus r))^\circ Q_{11}$ and $P_{12}((V, W)\oplus\mathcal{E}; r)^\circ Q_{12}$, we can infer that $(\nu\widetilde{c}.(P_{11}|Q_{11}), \nu\widetilde{c}.(P_{12}|Q_{12})) \in (\mathcal{E}; r)^\circ$ and we are done.
2. **Case** REACT-R: $C = C_1 \mid C_2$
   Similar.

Lemma B.20. [Reduction and environmental bisimulation up-to context]
Let $\mathcal{Y}$ be an environmental bisimulation up-to context. If $P\ \mathcal{Y}^\star_{\mathcal{E};r}\ Q$ and $P \rightarrow P'$ then there is a Q' such that $Q \Rightarrow Q'$ and $P'\ \mathcal{Y}^-_{\mathcal{E};r}\ Q'$.

Proof. Suppose $P\ \mathcal{Y}^\star_{\mathcal{E};r}\ Q$, therefore for some P_0, P_1, Q_0, Q_1, $\mathcal{E}'$, r', $\widetilde{c}$, $\widetilde{d}$ we have $P \equiv \nu\widetilde{c}.(P_0 \mid P_1)$, $Q \equiv \nu\widetilde{d}.(Q_0 \mid Q_1)$, $r \subseteq r'$, $\{\widetilde{c}\} \cap fn(r, \mathcal{E}.1) = \{\widetilde{d}\} \cap fn(r, \mathcal{E}.2) = \emptyset$, $\mathcal{E} \subseteq (\mathcal{E}'; r')^\star$, $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$ and $P_1\ (\mathcal{E}'; r')^\circ\ Q_1$.

We are going to analyse all the possible reduction transitions.

1. **Case**: $P \xrightarrow{\tau} P'$. We have four cases for the transitions of $\nu\widetilde{c}.(P_0 \mid P_1)$:

(a) **Subcase** $P_0 \xrightarrow{\tau} P_0'$

By $P_0 \mathcal{Y}_{\mathcal{E}';r'} Q_0$, we have that $Q_0 \Rightarrow Q_0'$ and $P_0' \mathcal{Y}^\star_{\mathcal{E}';r'} Q_0'$. Therefore, $\nu\widetilde{c}.(P_0 \mid P_1) \xrightarrow{\tau} \nu\widetilde{c}.(P_0' \mid P_1) \equiv \nu\widetilde{c},\widetilde{c}_i.(P_{00}' \mid P_{01}' \mid P_1) \equiv P'$ since $\widetilde{c}_i$ does not appear in P_1. Also, $\nu\widetilde{d}.(Q_0 \mid Q_1) \Rightarrow \nu\widetilde{d}.(Q_0' \mid Q_1) \equiv \nu\widetilde{d},\widetilde{d}_i.(Q_{00}' \mid Q_{01}' \mid Q_1) = Q'$ since $\widetilde{d}_i$ does not appear in Q_1. We have $\widetilde{c}_j$, $\widetilde{d}_j$, P_{00}, Q_{00}, P_{01}, Q_{01}, r'' and $\mathcal{E}''$ such that

- $P_0' \equiv \nu\widetilde{c}_i.(P_{00} \mid P_{01})$, $Q_0' \equiv \nu\widetilde{d}_i.(Q_{00} \mid Q_{01})$,
- $\widetilde{c}_i \notin fn(r', \mathcal{E}'.1)$, $\widetilde{d}_i \notin fn(r', \mathcal{E}'.2)$,
- $P_{00} \mathcal{Y}_{\mathcal{E}'';r''} Q_{00}$, $P_{01} (\mathcal{E}''; r'')^\circ Q_{01}$,
- $\mathcal{E}' \subseteq (\mathcal{E}''; r'')^\star$, $r' \subseteq r''$,

It then holds that $\mathcal{E} \subseteq (\mathcal{E}''; r'')^\star$, $r \subseteq r''$ and that $\widetilde{c},\widetilde{c}_i \notin fn(r, \mathcal{E}.1)$, $\widetilde{d},\widetilde{d}_i \notin fn(r, \mathcal{E}.2)$. Also, $P_1' (\mathcal{E}''; r'')^\circ Q_1'$ hence $(P_{01} \mid P_1') (\mathcal{E}''; r'')^\circ (Q_{01} \mid Q_1')$. Therefore $P' \mathcal{Y}^\star_{\mathcal{E};r} Q'$, hence $P' \mathcal{Y}^-_{\mathcal{E};r} Q'$.

(b) **Subcase** $P_1 \xrightarrow{\tau} P_1'$

Since $(P_1, Q_1) \in (\mathcal{E}'; r')^\circ$, then necessarily $P_1 \xrightarrow{\tau} P_1'$ implies that the context reduces, and Q_1 can do the same derivation. Therefore, by Lemmas B.18 and B.19 either $(P_1, Q_1) = (C[\widetilde{M}], C[\widetilde{N}]) \in (\mathcal{E}'; r')^\circ$ and $(P_1', Q_1') = (C'[\widetilde{M}], C'[\widetilde{N}]) \in (\mathcal{E}'; r')^\circ$ or $(P_1, Q_1) = (C_2[\text{`}\widetilde{A}, \text{`}A], C_2[\text{`}\widetilde{B}, \text{`}B]) = (C_p[\mathit{run}\text{`}\widetilde{A}, \mathit{run}\text{`}A], C_p[\mathit{run}\text{`}\widetilde{B}, \mathit{run}\text{`}B])$ and $(P_1', Q_1') = (C_p[\mathit{run}\text{`}\widetilde{A}, A], C_p[\mathit{run}\text{`}\widetilde{B}, B])$ with $(\text{`}A, \text{`}B) \in \mathcal{E}'$ and $(A, B) \notin (\mathcal{E}'; r')^\circ$. In both cases, we have $P' \mathcal{Y}^-_{\mathcal{E};r} Q'$ and we are done.

(c) **Subcase** $P_0 \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle V\rangle} P_0' \quad P_1 \xrightarrow{a(V)} P_1' \quad \{\widetilde{c}_1\} \cap fn(r', \mathcal{E}'.1) = \emptyset$

By $P_0 \mathcal{Y}_{\mathcal{E}';r'} Q_0$, we have $Q_0 \xRightarrow{\nu\widetilde{d}_1.\overline{a}\langle W\rangle} Q_0'$ and $P_0' \mathcal{Y}^\star_{(V,W)\oplus\mathcal{E}';r'} Q_0'$ with $\widetilde{d}_1 \notin fn(\mathcal{E}'.2, r')$ free in W. Also, since $P_1 \xrightarrow{a(V)} P_1'$ we have by Lemma B.4 that $Q_1 \xrightarrow{a(W)} Q_1'$ and $P_1' (\mathcal{E}' \cup \{(V, W)\}; r')^\circ Q_1'$. Therefore $\nu\widetilde{c}.(P_0 \mid P_1) \xrightarrow{\tau}\equiv \nu\widetilde{c},\widetilde{c}_1.(P_0' \mid P_1') \equiv P'$ and $\nu\widetilde{d}.(Q_0 \mid Q_1) \Rightarrow\equiv \nu\widetilde{d},\widetilde{d}_1.(Q_0' \mid Q_1') = Q'$. By $P_0' \mathcal{Y}^\star_{(V,W)\oplus\mathcal{E}';r'} Q_0'$ we have that

- $P_0'' = \nu\widetilde{c}_2.(P_{00} \mid P_{01})$, $Q_0'' = \nu\widetilde{d}_2.(Q_{00} \mid Q_{01})$,
- $\widetilde{c}_2 \notin fn(\mathcal{E}'.1, r')$, $\widetilde{d}_2 \notin fn(\mathcal{E}'.2, r')$,
- $P_{00} \mathcal{Y}_{\mathcal{E}'';r''} P_{00}$, $P_{01} (\mathcal{E}''; r'')^\circ Q_{01}$,
- $(V, W)\oplus\mathcal{E}' \subseteq (\mathcal{E}''; r'')^\star$, $r' \subseteq r''$.

We have $\nu\widetilde{c},\widetilde{c}_1.(P_0'' \mid P_1') \equiv \nu\widetilde{c},\widetilde{c}_1,\widetilde{c}_2.(P_{00} \mid P_{01} \mid P_1')$, $\nu\widetilde{d},\widetilde{d}_1.(Q_0'' \mid Q_1') \equiv \nu\widetilde{d},\widetilde{d}_1,\widetilde{d}_2.(Q_{00} \mid Q_{01} \mid Q_1')$, $\mathcal{E} \subseteq (\mathcal{E}''; r'')^\star$, $r \subseteq r''$, hence $(P_{01} \mid P_1') (\mathcal{E}''; r'')^\circ (Q_{01} \mid Q_1')$, as well as $\widetilde{c},\widetilde{c}_1,\widetilde{c}_2 \notin fn(\mathcal{E}.1, r)$, $\widetilde{d},\widetilde{d}_1,\widetilde{d}_2 \notin fn(\mathcal{E}.2, r)$. We thus conclude that $P \mathcal{Y}^-_{\mathcal{E};r} Q$.

(d) **Subcase** $P_0 \xrightarrow{a(V)} P_0' \quad P_1 \xrightarrow{\nu\widetilde{s}.\overline{a}\langle V\rangle} P_1'$

By Lemma B.5, we have for $\widetilde{s} \notin fn(r', \mathcal{E}')$ that $Q_1 \xrightarrow{\nu\widetilde{s}.\overline{a}\langle W\rangle} Q_1'$ and $V (\mathcal{E}'; \widetilde{s}\oplus r')^\star W$, as well as $P_1' (\mathcal{E}'; \widetilde{s}\oplus r')^\circ Q_1'$. Using freshness of $\widetilde{s}$, we have by clause 5 of the bisimulation up-to context $P_0 \mathcal{Y}_{\mathcal{E}';\widetilde{s}\oplus r'} Q_0$, hence $P_0' \mathcal{Y}^\star_{\mathcal{E}';\widetilde{s}\oplus r'} Q_0'$ for some $Q_0 \xRightarrow{a(W)} Q_0'$, i.e.

- $P_0' = \nu\widetilde{c}_1.(P_{00} \mid P_{01})$, $Q_0' = \nu\widetilde{d}_1.(Q_{00} \mid Q_{01})$,

– $\widetilde{c}_1 \notin fn(\mathcal{E}'.1, \widetilde{s}\oplus r')$, $\widetilde{d}_1 \notin fn(\mathcal{E}'.2, \widetilde{s}\oplus r')$,
– $P_{00}\, \mathcal{Y}_{\mathcal{E}'';r''}\, Q_{00}$, $P_{01}\, (\mathcal{E}'';r'')^\circ\, Q_{01}$,
– $\mathcal{E}' \subseteq (\mathcal{E}'';r'')^\star$, $\widetilde{s}\oplus r' \subseteq r''$.

Then, $\widetilde{c}, \widetilde{s}, \widetilde{c}_1 \notin fn(\mathcal{E}.1, r)$ and $\widetilde{d}, \widetilde{s}, \widetilde{d}_1 \notin fn(\mathcal{E}.2, r)$, $\nu\widetilde{c}.(P_0 \mid P_1) \xrightarrow{\tau} \nu\widetilde{c}, \widetilde{s}.(P_0' \mid P_1') \equiv \nu\widetilde{c}, \widetilde{s}, \widetilde{c}_1.(P_{00} \mid P_{01} \mid P_1') = P'$, and $\nu\widetilde{d}.(Q_0 \mid Q_1) \Rightarrow \nu\widetilde{d}, \widetilde{s}.(Q_0' \mid Q_1') \equiv \nu\widetilde{d}, \widetilde{s}, \widetilde{d}_1.(Q_{00} \mid Q_{01} \mid Q_1') = Q'$. Also $r \subseteq r''$ and $\mathcal{E} \subseteq (\mathcal{E}'';r'')^\star$, hence $(P_{01} \mid P_1')\, (\mathcal{E}'';r'')^\circ\, (Q_{01} \mid Q_1')$. As a result, $P'\, \mathcal{Y}^-_{\mathcal{E};r}\, Q'$.

2. **Case**: Q reduces.
Conversely.

Lemma B.21. [*run*-expanded output with spawning]
Suppose that $\nu\widetilde{c}.(P_0 \mid l[P_1])\, \mathcal{Y}^\star_{\mathcal{E};r}\, \nu\widetilde{d}.(Q_0 \mid l[Q_1])$ *for an environmental bisimulation up-to context* $\mathcal{Y}$ *with* $l \in r$ *and that* $P_1 \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M\rangle} P_1'$ *is minimal with respect to* $P_1^- \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M^-\rangle} P_1'^-$*, (so* $\nu\widetilde{c}.(P_0 \mid l[P_1]) \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M\rangle} \nu\widetilde{c}'.(P_0 \mid l[P_1'])$ *is minimal with respect to* $\nu\widetilde{c}.(P_0^- \mid l[P_1^-]) \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M^-\rangle} \nu\widetilde{c}'.(P_0^- \mid l[P_1'^-])$*). Then* $\nu\widetilde{d}.(Q_0 \mid l[Q_1]) \overset{\nu\widetilde{d}_0.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d}'.(Q_0' \mid l[Q_1'])$*, and* $\nu\widetilde{c}_r.P_0\, \mathcal{Y}^-_{(M,N)\oplus(`P_1',`Q_1')\oplus\mathcal{E};r}\, \nu\widetilde{d}_r.Q_0'$ *with* $\{\widetilde{c}'\} = \{\widetilde{c}\} \setminus fn(M)$*,* $\{\widetilde{c}_r\} = \{\widetilde{c}'\} \setminus fn(P_1')$*,* $\{\widetilde{d}_r\} = \{\widetilde{d}'\} \setminus fn(Q_1')$*, and* $\{\widetilde{c}', \widetilde{c}_0\} \cap fn(\mathcal{E}.1, r) = \{\widetilde{d}', \widetilde{d}_0\} \cap fn(\mathcal{E}.2, r) = \{d_0\} \cap \{\widetilde{d}'\} = \{c_0\} \cap \{\widetilde{c}'\} = \emptyset$*.*

Proof. By induction on n.

– **Case** $n = 0$
Immediate by Lemma B.6 and the fact that $\mathcal{Y}^\star \subseteq \mathcal{Y}^-$.
– **Case** $n > 0$
By Lemma B.20 and Lemma B.12, we have two possible subcases preserving minimality after the first *run*-transition of $\nu\widetilde{c}.(P_0 \mid l[P_1]) \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M\rangle} \nu\widetilde{c}'.(P_0 \mid l[P_1'])$.
  • **Subcase** $\nu\widetilde{c}.(P_0 \mid l[P_1]) \xrightarrow{run} \nu\widetilde{c}.(P_0 \mid l[P_1''])$, $\nu\widetilde{d}.(Q_0 \mid l[Q_1]) \Rightarrow \nu\widetilde{d}''.(Q_0'' \mid l[Q_1''])$ and $\nu\widetilde{c}.(P_0 \mid l[P_1''])\, \mathcal{Y}^\star_{\mathcal{E};r}\, \nu\widetilde{d}''.(Q_0'' \mid l[Q_1''])$.
  As $\nu\widetilde{c}.(P_0 \mid l[P_1'']) \overset{run^{n-1}}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M\rangle} \nu\widetilde{c}'.(P_0 \mid l[P_1'])$ is still minimal with respect to $\nu\widetilde{c}.(P_0^- \mid l[P_1^-]) \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M^-\rangle} \nu\widetilde{c}'.(P_0^- \mid l[P_1'^-])$, we can apply the induction hypothesis and get the desired results.
  • **Subcase** $\nu\widetilde{c}.(P_0 \mid l[P_1]) \xrightarrow{run} \nu\widetilde{c}.(P_0 \mid l[P_1''])$, $\nu\widetilde{d}.(Q_0 \mid l[Q_1]) \Rightarrow \nu\widetilde{d}''.(Q_0'' \mid l[Q_1''])$ and $\nu\widetilde{c}.(P_0 \mid l[P_1''])\, \mathcal{Y}^-_{\mathcal{E};r}\, \nu\widetilde{d}''.(Q_0'' \mid l[Q_1''])$, with
  $(P_1, Q_1) = (C_p[run(\widetilde{M}), run`A], C_p[run(\widetilde{N}), run`B])$ and
  $(P_1'', Q_1'') = (C_p[run(\widetilde{M}), A], C_p[run(\widetilde{N}), B])$ with (A, B) in redex position (i.e. $(`A, `B) \in \mathcal{E}'$ and $(A, B) \notin (\mathcal{E}';r')^\circ$ for some $\mathcal{E}'$, r' such that $\mathcal{E} \subseteq (\mathcal{E}';r')^\star$, $\nu\widetilde{c}.(P_0 \mid l[P_1]) \equiv \nu\widetilde{x}.(P_A \mid P_B)$, $\nu\widetilde{d}.(Q_0 \mid l[Q_1]) \equiv \nu\widetilde{y}.(Q_A \mid Q_B)$, $P_A\, \mathcal{Y}_{\mathcal{E}';r'}\, Q_A$, $(P_B, Q_B) \in (r';\mathcal{E}')^\circ$, $r \subseteq r'$, $\widetilde{x} \cap fn(\mathcal{E}.1, r) = \widetilde{y} \cap fn(\mathcal{E}.2, r) = \emptyset$.
  By $\nu\widetilde{c}.(P_0^- \mid l[P_1^-]) \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M^-\rangle} \nu\widetilde{c}'.(P_0^- \mid l[P_1'^-])$, $P_1 = C_p[run(\widetilde{M}), run`A]$, $P_1'' = C_p[run(\widetilde{M}), A]$ and $P_1^- \leq P_1'$, we know that there is a *run*-erasure

$A^- \leq A$ such that A^- is in redex position in P_1 and that $A \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M\rangle} A'$ is minimal with respect to $A^- \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M^-\rangle} A'^-$. Using Lemma B.16 (to add a fresh name l), Lemma B.17 and up-to context (to work with P_0 and Q_0 instead of P_A and Q_A) and environment (to work with $\mathcal{E}$ and r instead of $\mathcal{E}'$, r') techniques, we have $P_0 \mid l[A] \; \mathcal{Y}^{\star}_{\mathcal{E};l\oplus r} \; Q_0 \mid l[B]$ and, by Lemma B.13, $P_0 \mid l[A] \overset{run^{n-1}}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M\rangle} P_0 \mid l[A']$ minimal with respect to $P_0^- | l[A^-] \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M^-\rangle} P_0^- \mid l[A'^-]$. Applying the induction hypothesis, we get $Q_0'' \mid l[B] \overset{\nu\widetilde{d}_1.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d}_n.(Q_0' \mid l[B'])$, and $P_0 \; \mathcal{Y}^{-}_{(M,N)\oplus(`A',`B')\oplus\mathcal{E};l\oplus r} \; \nu\widetilde{d}_{r1}.Q_0'$ with $\{\widetilde{d}_{r1}\} = \{\widetilde{d}_n\} \setminus fn(B)$, and $\{\widetilde{d}_n\} \cap fn(\mathcal{E}.2, l, r) = \{\widetilde{d}_1\} \cap \{\widetilde{d}_n\} = \emptyset$. Therefore, $Q_0'' \mid l[Q_1''] \overset{\nu\widetilde{d}_1.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d}_n.(Q_0' \mid l[Q_1'])$ for $Q_1' = Cp[run(\widetilde{M}), B']$ since the context C_p cannot bind names free in $\mathcal{E}'$, r', hence in $\mathcal{E}$, B, nor in N. Then $\nu\widetilde{d}.(Q_0 \mid l[Q_1]) \overset{\nu\widetilde{d}_0.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d'}.(Q_0' \mid l[Q_1'])$ with $\{\widetilde{d'}\} = \{\widetilde{d}_n\} \cup \{\{\widetilde{d}\} \setminus \{\{\widetilde{d}_0\} \setminus \{\widetilde{d}_1\}\}\}$. And also, $P_0 \; \mathcal{Y}^{-}_{(M,N)\oplus(`A',`B')\oplus\mathcal{E};l\oplus r} \; \nu\widetilde{d}_{r1}.Q_0'$ implies $P_0 \; \mathcal{Y}^{-}_{(M,N)\oplus(`P_1',`Q_1')\oplus\mathcal{E};r} \; \nu\widetilde{d}_{r1}.Q_0'$ up-to environment to remove l and build P_1' and Q_1', and $\nu\widetilde{c}_r.P_0 \; \mathcal{Y}^{-}_{(M,N)\oplus(`P_1',`Q_1')\oplus\mathcal{E};r} \; \nu\widetilde{d}_r.Q_0'$ up-to restriction for $\{\widetilde{c}_r\} = \{\widetilde{c'}\} \setminus fn(P_1')$ and $\{\widetilde{d}_r\} = \{\widetilde{d'}\} \setminus fn(Q_1')$.

Corollary B.22. [*run*-expanded output]
Suppose that $\nu\widetilde{c}.(P_0 \mid P_1) \; \mathcal{Y}^{\star}_{\mathcal{E};r} \; \nu\widetilde{d}.(Q_0 \mid Q_1)$ for an environmental bisimulation up-to context $\mathcal{Y}$ with $P_0 \; \mathcal{Y}_{\mathcal{E}';r'} \; Q_0$, $(P_1, Q_1) \in (\mathcal{E}'; r')^{\circ}$, $\{\widetilde{c}\} \cap fn(\mathcal{E}.1, r) = \{\widetilde{d}\} \cap fn(\mathcal{E}.2, r) = \emptyset$, and that $\nu\widetilde{c}.(P_0 \mid P_1) \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M\rangle} \nu\widetilde{c'}.(P_0' \mid P_1')$ is minimal with respect to $\nu\widetilde{c}.(P_0^- \mid P_1^-) \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M^-\rangle} \nu\widetilde{c'}.(P_0'^- \mid P_1'^-)$. Then $\nu\widetilde{d}.(Q_0 \mid Q_1) \overset{\nu\widetilde{d}_0.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d'}.(Q_0' \mid Q_1')$, and $\nu\widetilde{c'}.(P_0' \mid P_1') \; \mathcal{Y}^{-}_{(M,N)\oplus\mathcal{E};r} \; \nu\widetilde{d'}.(Q_0' \mid Q_1')$ with $\{\widetilde{d'}\} \cap fn(\mathcal{E}.2, r, N) = \{d_0\} \cap \{\widetilde{d'}\} = \emptyset$.

Proof. By induction on n.

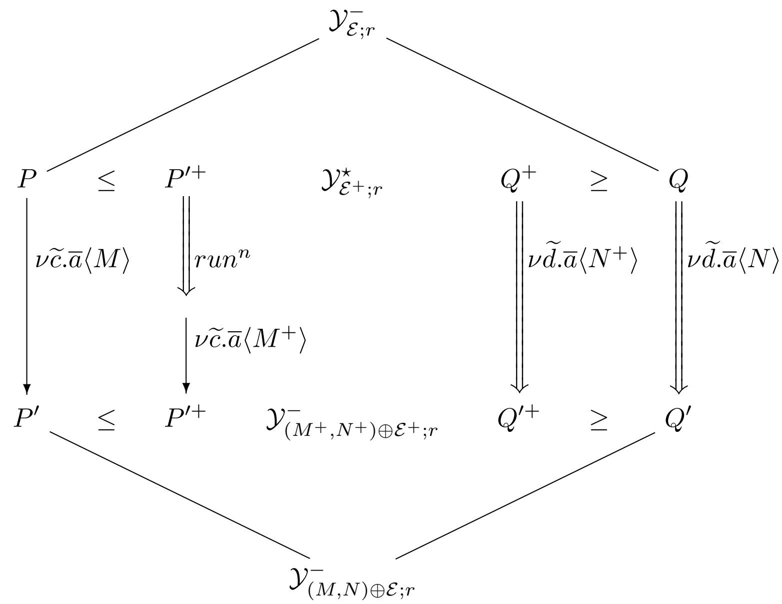
- **Case** $n = 0$
  Immediate by Lemma B.6 and the fact that $\mathcal{Y}^{\star} \subseteq \mathcal{Y}^{-}$.
- **Case** $n > 0$
  By Lemma B.20 and Lemma B.12, we have two possible subcases preserving minimality after the first *run*-transition of $\nu\widetilde{c}.(P_0 \mid P_1) \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M\rangle} \nu\widetilde{c'}.(P_0' \mid P_1')$.
  - **Subcase** $P_0 \xrightarrow{run} P_0''$
    We have that $Q_0 \overset{\tau}{\Rightarrow} Q_0''$ and $P_0'' \; \mathcal{Y}^{\star}_{\mathcal{E}';r'} \; Q_0''$, and also that $P_0'' \overset{run^{n-1}}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M\rangle} P_0'$ is minimal with respect to $P_0^- \xrightarrow{\nu\widetilde{c}_1.\overline{a}\langle M^-\rangle} P_0'^-$. Thus, we can apply the induction hypothesis and get $Q_0'' \overset{\nu\widetilde{d}_1.\overline{a}\langle N\rangle}{\Longrightarrow} Q_0'$ as well as $P_0' \; \mathcal{Y}^{-}_{(M,N)\oplus\mathcal{E}';r'} \; Q_0'$. Therefore, $Q_0 \mid Q_1 \overset{\nu\widetilde{d}_1.\overline{a}\langle N\rangle}{\Longrightarrow} Q_0' \mid Q_1$, hence $\nu\widetilde{d}.(Q_0 \mid Q_1) \overset{\nu\widetilde{d}_0.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d'}.(Q_0' \mid Q_1)$. Also, $P_0' \mid P_1 \; \mathcal{Y}^{-}_{(M,N)\oplus\mathcal{E};r} \; Q_0' \mid Q_1$ up-to environment for using $\mathcal{E}$ and r instead

of $\mathcal{E}'$ and r' and context for spawning P_1 and Q_1, and finally $\nu\widetilde{c}'.(P'_0 \mid P_1) \, \mathcal{Y}^-_{(M,N)\oplus\mathcal{E};r} \, \nu\widetilde{d}'''.(Q'_0 \mid Q_1)$ up-to restriction for $\{\widetilde{c}'\} = \{\widetilde{c}\} \setminus fn(M)$, $\{\widetilde{d}'''\} \supseteq \{\widetilde{d}\} \setminus fn(N)$, $\{\widetilde{d}'\} \notin fn(\mathcal{E}.2, N, r)$, and $d' = d'''$.

- **Subcase** $P_1 \xrightarrow{run} P'_1$
  Using Lemma B.16 to add a fresh name l and the fact that $(P_1, Q_1) \in (\mathcal{E}'; r')^\circ$, we have $\nu\widetilde{c}.(P_0 \mid l[P_1]) \, \mathcal{Y}^\star_{\mathcal{E};l\oplus r} \, Q_0 \mid l[Q_1]$. As $\nu\widetilde{c}.(P_0 \mid l[P_1]) \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M\rangle} \nu\widetilde{c}.(P_0 \mid l[P'_1])$ is minimal with respect to $\nu\widetilde{c}.(P^-_0 \mid l[P^-_1]) \xrightarrow{\nu\widetilde{c}_0.\overline{a}\langle M^-\rangle} \nu\widetilde{c}'.(P^-_0 \mid l[P'^-_1])$, we can use Lemma B.21 and have $\nu\widetilde{d}.(Q_0 \mid l[Q_1]) \overset{\nu\widetilde{d}_0.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d}'.(Q'_0 \mid l[Q'_1])$, hence $\nu\widetilde{d}.(Q_0 \mid Q_1) \overset{\nu\widetilde{d}_0.\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d}'.(Q'_0 \mid Q'_1)$ and also $\nu\widetilde{c}_r.P_0 \, \mathcal{Y}^-_{(M,N)\oplus(`P'_1,`Q'_1)\oplus\mathcal{E};l\oplus r} \, \nu\widetilde{d}_r.Q'_0$ with $\{\widetilde{c}_r\} = \{\widetilde{c}'\} \setminus fn(M)$, $\{\widetilde{d}_r\} \subseteq \{\widetilde{d}'\} \setminus fn(N)$, $\{\widetilde{d}'\} \cap fn(\mathcal{E}.2, N, r) = \emptyset$. Therefore, $\nu\widetilde{c}'.(P_0 \mid P'_1) \, \mathcal{Y}^-_{(M,N)\oplus\mathcal{E};r} \, \nu\widetilde{d}'.(Q'_0 \mid Q'_1)$ up-to context, environment and restriction.

Corollary B.23. [Output preserves *run*-erased environmental bisimulation up-to context]
For any environmental bisimulation up-to context $\mathcal{Y}$, if $P \, \mathcal{Y}^-_{\mathcal{E};r} \, Q$ and $P \xrightarrow{\nu\widetilde{c}.\overline{a}\langle M\rangle} P'$ with $a \in r$ and $\widetilde{c} \notin fn(\mathcal{E}.1, r)$, then there is a Q' such that $Q \overset{\nu\widetilde{d}.\overline{a}\langle N\rangle}{\Longrightarrow} Q'$ with $\widetilde{d} \notin fn(\mathcal{E}.2, r)$ and $P' \, \mathcal{Y}^-_{(M,N)\oplus\mathcal{E};r} \, Q'$. The converse on Q's transition holds too.

Proof. By $\mathcal{Y}^-$'s definition, we know there are P^+, Q^+ and $\mathcal{E}^+$ such that $P^+ \, \mathcal{Y}^\star_{\mathcal{E}^+;r} \, Q^+$. Since $P \xrightarrow{\nu\widetilde{c}.\overline{a}\langle M\rangle} P'$, there is a minimal output transition $P^+ \overset{run^n}{\Longrightarrow} \xrightarrow{\nu\widetilde{c}.\overline{a}\langle M^+\rangle} P'^+$. By Lemma B.22, we have $Q^+ \overset{\nu\widetilde{c}.\overline{a}\langle N^+\rangle}{\Longrightarrow} Q'^+$ and $P'^+ \, \mathcal{Y}^-_{(M^+,N^+)\oplus\mathcal{E}^+;r} \, Q'^+$ which implies by Corollary B.9 that Q can also weakly do an output transition $Q \overset{\nu\widetilde{c}.\overline{a}\langle N\rangle}{\Longrightarrow} Q'$, such that $Q' \leq Q'^+$ and $N \leq N^+$. By Corollary B.15, as $P' \leq P'^+$, $Q' \leq Q'^+$ and $(M,N)\oplus\mathcal{E} \leq (M^+,N^+)\oplus\mathcal{E}^+$, we have $P' \, \mathcal{Y}^-_{(M,N)\oplus\mathcal{E};r} \, Q'$ as desired. Visually, the following diagram holds.



The converse on Q's transitions is shown similarly.

Lemma B.24. [*run*-expanded input]
Suppose that $P\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q$ for an environmental bisimulation up-to context $\mathcal{Y}$ and that $P \stackrel{run^n}{\Longrightarrow} \xrightarrow{a(M)} P'$ is minimal with respect to $P^{-} \xrightarrow{a(M^{-})} P'^{-}$. Then for all N such that $(M,N) \in (\mathcal{E};r)^{\star}$, $Q \stackrel{a(N)}{\Longrightarrow} Q'$, and $P'\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q'$.

Proof. By induction on n.

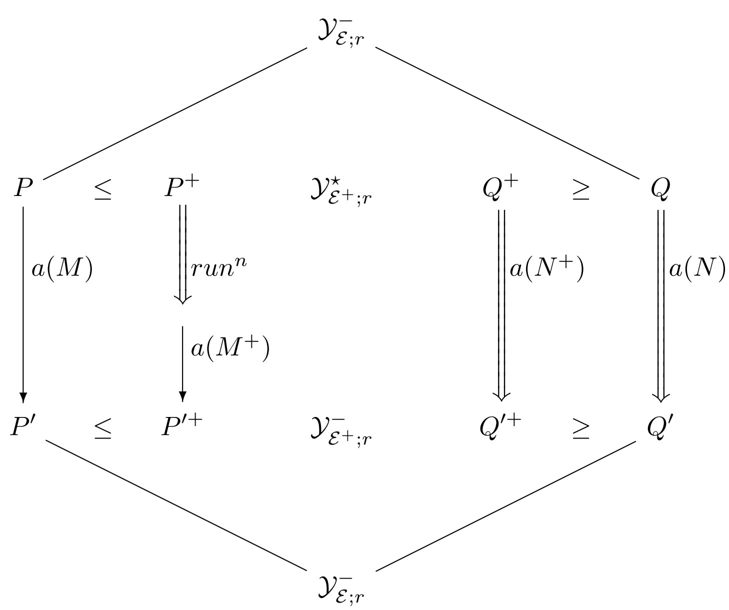
- **Case** $n = 0$
  Immediate by Lemma B.6 and the fact that $\mathcal{Y}^{\star} \subseteq \mathcal{Y}^{-}$.
- **Case** $n > 0$
  By Lemma B.20 and Lemma B.12, we have two possible subcases preserving minimality after the first *run*-transition of $P \stackrel{run^n}{\Longrightarrow} \xrightarrow{a(M)} P'$.
  - **Subcase** $P \xrightarrow{run} P''$, $Q \stackrel{\tau}{\Rightarrow} Q''$, $P''\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q''$

    We have that $P'' \stackrel{run^{n-1}}{\Longrightarrow} \xrightarrow{a(M)} P'$ is still minimal, hence we can apply the induction hypothesis, and we are done.
  - **Subcase** $P \xrightarrow{run} P''$, $Q \stackrel{\tau}{\Rightarrow} Q''$, $P''\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q''$ with $P'' = \nu\widetilde{c}.(P_0 \mid P_1)$ and $Q'' = \nu\widetilde{d}.(Q_0 \mid Q_1)$, $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, $(P_1, Q_1) = (C_p[\mathit{run}(\widetilde{M}), A], C_p[\mathit{run}(\widetilde{N}), B]) \in (\mathcal{E}';r')^{-} \setminus (\mathcal{E}';r')^{\circ}$, $((\widetilde{M};`A), (\widetilde{N};`B)) \in \mathcal{E}'$, $\{\widetilde{c}\} \cap \mathit{fn}(\mathcal{E}.1, r) = \{\widetilde{d}\} \cap \mathit{fn}(\mathcal{E}.2, r) = \emptyset$.
    Using Lemma B.16 (to add a fresh l to r) and clause 4 of environmental bisimulations up-to context, we have $P_0 \mid l[A]\ \mathcal{Y}^{\star}_{\mathcal{E}';l\oplus r'}\ Q_0 \mid l[B]$. Using an argument similar to the one in Lemma B.21, case 2, subcase 2, we know that we can apply

the induction hypothesis to minimal transition $P_0 \mid l[A] \xRightarrow{run^{n-1}} \xrightarrow{a(M)} P_0 \mid l[A']$. We obtain $Q_0 \mid l[B] \xRightarrow{a(N)} \nu\widetilde{d}_n(Q_0'|l[B'])$ and $P_0 \mid l[A']\ \mathcal{Y}^-_{\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_n.(Q_0' \mid l[B'])$. By Corollary B.23, we have, after an output to channel l, $P_0\ \mathcal{Y}^-_{(A',B')\oplus\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_r.Q_0'$, hence $P_0 \mid C_p[run(\widetilde{M}), A']\ \mathcal{Y}^-_{\mathcal{E};r}\ \nu\widetilde{d}_r.Q_0' \mid C_p[run(\widetilde{N}), B']$ up-to environment and context, and $\nu\widetilde{c}.(P_0 \mid C_p[run(\widetilde{M}), A'])\ \mathcal{Y}^-_{\mathcal{E};r}\ \nu\widetilde{d'}.(Q_0' \mid C_p[run(\widetilde{N}), B'])$, up-to structural congruence and restriction. And of course, we do have $Q \xRightarrow{a(N)} \nu\widetilde{d'}.(Q_0' \mid C_p[run(\widetilde{N}), B']))$ since the constraint on $\widetilde{d'}$ and on names bound by the context could not hinder the input.

Corollary B.25. [Input preserves *run*-erased environmental bisimulation up-to context]
For any environmental bisimulation up-to context $\mathcal{Y}$, if $P\ \mathcal{Y}^-_{\mathcal{E};r}\ Q$ and $P \xrightarrow{a(M)} P'$ with $a \in r$, then there is a Q' such that for all $(M,N) \in (\mathcal{E};r)^\star$, $Q \xRightarrow{a(N)} Q'$ and $P'\ \mathcal{Y}^-_{\mathcal{E};r}\ Q'$. The converse on Q's transitions holds too.

Proof. By $\mathcal{Y}^-$'s definition, we know there are P^+, Q^+ and $\mathcal{E}^+$ such that $P^+\ \mathcal{Y}^\star_{\mathcal{E}^+;r}\ Q^+$. Since $P \xrightarrow{a(M)} P'$, there is a minimal input transition $P^+ \xRightarrow{run^n} \xrightarrow{a(M^+)} P'^+$. By Lemma B.24, we have $Q^+ \xRightarrow{a(N^+)} Q'^+$ for any $(M^+, N^+) \in (\mathcal{E}^+;r)^\star$ and $P'^+\ \mathcal{Y}^-_{\mathcal{E}^+;r}\ Q'^+$ which implies by Corollary B.9 that Q can also weakly do an input transition $Q \xRightarrow{a(N)} Q'$ such that $Q' \le Q'^+$ for any $N \le N^+$, i.e. for any $(M,N) \in (\mathcal{E};r)^-$. By Corollary B.15, as $P' \le P'^+$, $Q' \le Q'^+$ and $\mathcal{E} \le \mathcal{E}^+$, we have $P'\ \mathcal{Y}^-_{\mathcal{E};r}\ Q'$ as desired. Visually, the following diagram holds.



The converse on Q's transitions is shown similarly.

Definition B.26. [Simple process]
We define the syntax of simple processes, a subset of processes, as

$$P_r ::= 0 \mid a.P_r \mid \overline{a}\langle `P_r\rangle.P_r \mid (P_r \mid P_r) \mid a[P_r] \mid !P_r \mid \mathit{run}(`P_r)$$

Definition B.27. [simple environmental bisimulation up-to context]
An environmental bisimulation up-to context $\mathcal{Y}$ is said to be simple if all of its environments contain only simple processes.

Lemma B.28. [*run*-expanded reduction for simple environmental bisimulation up-to context]
Suppose that $P\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q$ for a simple environmental bisimulation up-to context $\mathcal{Y}$, and that $P \stackrel{run^n}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} P'$ is minimal with respect to $P^- \stackrel{\tau}{\longrightarrow} P'^-$, then $Q \stackrel{\hat{\tau}}{\Rightarrow} Q'$, and $P'\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q'$.

Proof. By induction on n.

- **Case** $n = 0$
  Immediate by Lemma B.20.
- **Case** $n > 0$
  By Lemma B.20 and Lemma B.12, we have two possible subcases preserving minimality after the first *run*-transition of $P \stackrel{run^n}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} P'$.
  - **Subcase** $P \stackrel{run}{\longrightarrow} P''$, $Q \stackrel{\hat{\tau}}{\Rightarrow} Q''$, $P''\ \mathcal{Y}^{\star}_{\mathcal{E};r}\ Q''$
    We have that $P'' \stackrel{run^{n-1}}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} P'$ is still minimal with respect to $P^- \stackrel{\tau}{\longrightarrow} P'^-$, so we can apply the induction hypothesis, and we are done.
  - **Subcase** $P \stackrel{run}{\longrightarrow} P''$, $Q \stackrel{\hat{\tau}}{\Rightarrow} Q''$, $P''\ \mathcal{Y}^{-}_{\mathcal{E};r}\ Q''$ hold, and we have $P'' = \nu\widetilde{c}.(P_0 \mid P_1)$ and $Q'' = \nu\widetilde{d}.(Q_0 \mid Q_1)$ with $P_0\ \mathcal{Y}_{\mathcal{E}';r'}\ Q_0$, $(P_1, Q_1) = (C_p[\mathit{run}(\widetilde{M}), A], C_p[\mathit{run}(\widetilde{N}), B]) \in (\mathcal{E}';r')^- \backslash (\mathcal{E}';r')^\circ$, $((\widetilde{M}; `A), (\widetilde{N}, `B)) \in \mathcal{E}'$, $\{\widetilde{c}\} \cap \mathit{fn}(\mathcal{E}.1, r) = \{\widetilde{d}\} \cap \mathit{fn}(\mathcal{E}.2, r) = \emptyset$. Since we know $P^- \stackrel{\tau}{\longrightarrow} P'^-$ and that $P \stackrel{run^n}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} P'$ is minimal with respect to it, we can infer how P weakly reduces to P'. Let us analyse each possibility.
    - **Subsubcase** A reacts with P_0
      Using clause 5 to add a new name l to r and clause 4 to spawn A, we have $P_0 \mid l[A]\ \mathcal{Y}^{\star}_{\mathcal{E}';l\oplus r'}\ Q_0 \mid l[B]$. Using an argument similar to the one in Lemma B.21, case 2, subcase 2, we know that we can apply the induction hypothesis to minimal transition $P_0 \mid l[A] \stackrel{run^{n-1}}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} \nu\widetilde{c}_0.(P'_0 \mid l[A'])$. We obtain $Q_0 \mid l[B] \stackrel{\hat{\tau}}{\Rightarrow} \nu\widetilde{d}_0.(Q'_0 \mid l[B'])$ and $\nu\widetilde{c}_0.(P'_0 \mid l[A'])\ \mathcal{Y}^{-}_{\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_0.(Q'_0 \mid l[B'])$. Therefore, not only do we have $Q'' \stackrel{\hat{\tau}}{\Rightarrow} \nu\widetilde{d'}.(Q'_0 \mid C_p[\ N, B'])$ with $\{\widetilde{d'}\} = \{\widetilde{d}\} \cup \{\widetilde{d}_0\}$ since the context C_p could not bind names used in the reaction, but also by $\nu\widetilde{c}_0.(P'_0 \mid l[A'])\ \mathcal{Y}^{-}_{\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_0.(Q'_0 \mid l[B'])$ do we have $\nu\widetilde{c}_r.P'_0\ \mathcal{Y}^{-}_{(`A',B')\oplus\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_r.Q'_0$ by Corollary B.23, hence
      $\nu\widetilde{c}_r.P'_0 \mid C_p[\mathit{run}(\widetilde{M}), A']\ \mathcal{Y}^{-}_{(`A',B')\oplus\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_r.Q'_0 \mid C_p[\mathit{run}(\widetilde{N}), B']$ up-to context,
      $\nu\widetilde{c}_r.P'_0 \mid C_p[\mathit{run}(\widetilde{M}), A']\ \mathcal{Y}^{-}_{\mathcal{E};r}\ \nu\widetilde{d}_r.Q'_0 \mid C_p[\mathit{run}(\widetilde{N}), B']$ up-to environment,

$\nu\widetilde{c}_0.(P_0' \mid C_p[run(\widetilde{M}), A']) \,\mathcal{Y}^-_{\mathcal{E};r}\, \nu\widetilde{d}_0.(Q_0' \mid C_p[run(\widetilde{N}), B'])$ up-to restriction and structural congruence, and finally
$\nu\widetilde{c}'.(P_0' \mid C_p[run(\widetilde{M}), A']) \,\mathcal{Y}^-_{\mathcal{E};r}\, \nu\widetilde{d}'.(Q_0' \mid C_p[run(\widetilde{N}), B'])$ up-to restriction.

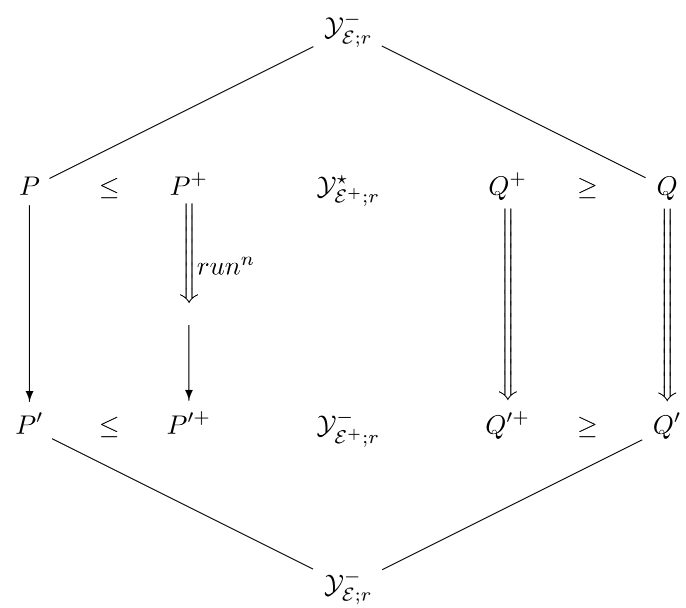
- $*$ **Subsubcase** A reduces alone
  Similarly, but with $\widetilde{c}' = \widetilde{c}$ and $P_0' = P_0$.
- $*$ **Subsubcase** A reacts with an A_i from $`\widetilde{A} = \widetilde{M}$ (or a *run*-erasure of it)
  Let $G = C[`A_i, \widetilde{M}']$ (resp. $H = C[`B_i, \widetilde{N}']$) be the process of P_1 (resp. Q_1) in redex position that contains A_i (resp. B_i) and reacts with the process containing A according to rule REACT-R or REACT-L. Then there is a process context C_p' such that $C_p[A, run(\widetilde{M})] = C_p'[A', G, run(\widetilde{M}'')]$. By clause 5 of environmental bisimulation to add a new name l and clause 4, we have $P_0 \mid l[A] \,\mathcal{Y}^\star_{\mathcal{E}';l\oplus r'}\, Q_0 \mid l[B]$. By Lemma B.16 to add a new name m to r and by up-to context, we have
  $P_0 \mid l[A] \mid m[G] \,\mathcal{Y}^\star_{\mathcal{E}';m\oplus l\oplus r'}\, Q_0 \mid l[B] \mid m[H]$.
  By the conditions on $\mathcal{E}'$, we know we have
  $P_0 \mid l[A] \mid m[G] \overset{run}{\Longrightarrow} \overset{\tau}{\longrightarrow} P_0 \mid l[A'] \mid m[G']$ with no name extruded.
  For the same reasons, when applying the induction hypothesis, we obtain
  $Q_0 \mid l[B] \mid m[H] \overset{\tau}{\Rightarrow} \nu\widetilde{d}_0.(Q_0' \mid l[B'] \mid m[H'])$ such that there is no bound name whose scope is only B' and H', as well as
  $P_0 \mid l[A'] \mid m[G'] \,\mathcal{Y}^-_{\mathcal{E}';m\oplus l\oplus r'}\, \nu\widetilde{d}_0.(Q_0' \mid l[B'] \mid m[H'])$.
  We can now passivate the contents of $l[\,]$ and $m[\,]$ to replace them up-to context, restriction and structural equivalence to get
  $P_0 \mid C_p'[A', G', run(\widetilde{M}'')] \,\mathcal{Y}^-_{\mathcal{E}';m\oplus l\oplus r'}\, \nu\widetilde{d}_0.(Q_0' \mid C_p'[B', H', run(\widetilde{N}'')])$
  and finally use up-to environment and restriction to obtain
  $\nu\widetilde{c}.(P_0 \mid C_p'[A', G', run(\widetilde{M}'')]) \,\mathcal{Y}^-_{\mathcal{E};r}\, \nu\widetilde{d}'.(Q_0' \mid C_p'[B', H', run(\widetilde{N}'')])$.
- $*$ **Subsubcase** A outputs and reacts with the context
  Let $C[\widetilde{M}']$ (resp. $C[\widetilde{N}']$) be the process of P_1 (resp. Q_1) in redex position that reacts with the process containing A according to rule REACT-R or REACT-L. Then there is another context C_p' such that $C_p[A, run(\widetilde{M})] = C_p'[A', C[\widetilde{M}'], run(\widetilde{M}'')]$. Using clause 5 to add a new name l to r and clause 4 to spawn A, we have
  $P_0 \mid l[A] \,\mathcal{Y}^\star_{\mathcal{E}';l\oplus r'}\, Q_0 \mid l[B]$.
  We can now apply Lemma B.22 to simulate A's output after *run*-transitions,
  $P_0 \mid l[A] \overset{run^{n-1}}{\Longrightarrow} \overset{\overline{a}\langle M\rangle}{\longrightarrow} P_0 \mid l[A']$, and we obtain
  $Q_0 \mid l[B] \overset{\nu\widetilde{d}_1\overline{a}\langle N\rangle}{\Longrightarrow} \nu\widetilde{d}_0.(Q_0' \mid l[B'])$, with $\widetilde{d}_1$'s scope encompassing Q_0, and $P_0 \mid l[A'] \,\mathcal{Y}^-_{(M,N)\oplus\mathcal{E}';l\oplus r'}\, \nu\widetilde{d}_0.(Q_0' \mid l[B'])$ such that (M, N) belong (modulo *run*-erasure) to some $\mathcal{E}''$ with no bound names or to its context closure $(\mathcal{E}''; r'')^\star$. We passivate the contents of $l[\,]$ and have
  $P_0 \,\mathcal{Y}^-_{(`A',`B')\oplus(M,N)\oplus\mathcal{E}';l\oplus r'}\, \nu\widetilde{d}_r.Q_0'$. By the restrictions on $\mathcal{E}'$, we also know $(`A', `B')$ belong (modulo *run*-erasure) to $\mathcal{E}''$ or to its context closure. By Lemma B.4, we have that $C[\widetilde{M}] \xrightarrow{a(M)} G$ and $C[\widetilde{N}] \xrightarrow{a(N)} H$ with $(G, H) \in ((M, N)\oplus\mathcal{E}'; r')^\circ$. Up-to context, we build

$P_0 \mid C'_p[A', G, run(\widetilde{M''})]\ \mathcal{Y}^-_{(`A',`B')\oplus(M,N)\oplus\mathcal{E}';l\oplus r'}\ \nu\widetilde{d}_r.Q'_0 \mid C'_p[B', H, run(\widetilde{N''})]$
and then up-to environment and restriction
$\nu\widetilde{c}.(P_0 \mid C'_p[A', G, run(\widetilde{M})])\ \mathcal{Y}^-_{\mathcal{E};r}\ \nu\widetilde{d'}.(Q'_0 \mid C_p[B', H, run(\widetilde{N})])$.

- **Subsubcase** A inputs and reacts with the context
Suppose the context outputs a process M (resp. N) and extrudes names $\widetilde{x}$ by means of a process $C_o[\widetilde{M'}]$ (resp. $C_o[\widetilde{N'}]$) such that $C_o[\widetilde{M'}] \xrightarrow{\nu\widetilde{x}.\overline{a}\langle M\rangle} C'_o[\widetilde{M'}]$ (resp. $C_o[\widetilde{N'}] \xrightarrow{\nu\widetilde{x}.\overline{a}\langle N\rangle} C'_o[\widetilde{N'}]$ by Lemma B.5). Using clause 5 several times to add a new names l and $\widetilde{x}$ to r and clause 4 to spawn A, we have $P_0 \mid l[A]\ \mathcal{Y}^\star_{\mathcal{E}';l\oplus\widetilde{x}\oplus r'}\ Q_0 \mid l[B]$.
We can now apply Lemma B.24 to trigger A's input of M (resp. B's input of N) where $\widetilde{x}$ is free, after some *run*-transitions and we obtain
$P_0 \mid l[A']\ \mathcal{Y}^-_{\mathcal{E}';l\oplus\widetilde{x}\oplus r'}\ \nu\widetilde{d}_0.(Q'_0 \mid l[B'])$.
We passivate the content of $l[\,]$, obtaining
$P_0\ \mathcal{Y}^-_{(`A',`B')\oplus\mathcal{E}';l\oplus\widetilde{x}\oplus r'}\ \nu\widetilde{d}_r.Q'_0$.
Because of the constraints on input from elements of $\mathcal{E}'$, we know that $\widetilde{x}$ is not free in A' nor B'. We remove l and $\widetilde{x}$ up-to environment from the known names and then replace A' and B' up-to context, giving
$P_0 \mid C'_p[A', run(\widetilde{M})]\ \mathcal{Y}^-_{\mathcal{E}';r'}\ \nu\widetilde{d}_r.Q'_0 \mid C'_p[B', run(\widetilde{N})]$.
Notice that, since $\widetilde{x}$ does not appear in A', B', r' nor $\mathcal{E}'$ but could still be free in $C'_o[\widetilde{M'}]$, we have the context C'_p bind it properly around A' and $C'_o[\widetilde{M'}]$ (resp. B' and $C'_o[\widetilde{N'}]$) as required by the reaction transition. Finally, up-to environment and restriction, we have
$\nu\widetilde{c}.(P_0 \mid C'_p[A', run(\widetilde{M})])\ \mathcal{Y}^-_{\mathcal{E};r}\ \nu\widetilde{d'}.(Q'_0 \mid C_p[B', run(\widetilde{N})])$ and we are done.

Corollary B.29. [Reduction preserves *run*-erased simple environmental bisimulation up-to context]
For any simple environmental bisimulation up-to context $\mathcal{Y}$, if $P\ \mathcal{Y}^-_{\mathcal{E};r}\ Q$ and $P \xrightarrow{\tau} P'$, then there is a Q' such that $Q \overset{\tau}{\Rightarrow} Q'$ and $P'\ \mathcal{Y}^-_{\mathcal{E};r}\ Q'$. The converse on Q's transitions holds too.

Proof. By $\mathcal{Y}^-$'s definition, we know there are P^+, Q^+ and $\mathcal{E}^+$ such that $P^+\ \mathcal{Y}^\star_{\mathcal{E}^+;r}\ Q^+$. Since $P \xrightarrow{\tau} P'$, there is a minimal reduction transition $P^+ \overset{run^n}{\Longrightarrow}\xrightarrow{\tau} P'^+$. By Lemma B.28, we have $Q^+ \overset{\tau}{\Rightarrow} Q'^+$ and $P'^+\ \mathcal{Y}^-_{\mathcal{E}^+;r}\ Q'^+$ which implies by Corollary B.9 that Q can also weakly reduce to some Q' such that $Q' \leq Q'^+$. By Corollary B.15, as $P' \leq P'^+$, $Q' \leq Q'^+$ and $\mathcal{E} \leq \mathcal{E}^+$, we have $P'\ \mathcal{Y}^-_{\mathcal{E};r}\ Q'$ as desired. Visually, the following diagram holds.



The converse on Q's transitions is shown similarly.

Theorem B.30. [Soundness of simple environmental bisimulation up-to context] *If $\mathcal{Y}$ is a simple environmental bisimulation up-to context, then $\mathcal{Y}^-$ is included in bisimilarity.*

Proof. Let $\mathcal{X} = \{(r, \mathcal{E}, P, Q) \mid P \ \mathcal{Y}^-_{\mathcal{E};r} \ Q\}$ and let us prove that $\mathcal{X}$ verifies each clause of environmental bisimulation.

1. By Corollary B.29, whenever $P \xrightarrow{\tau} P'$, we have a Q' such that $Q \overset{\tau}{\Rightarrow} Q'$ and $P' \ \mathcal{Y}^-_{\mathcal{E};r} \ Q'$, i.e. $P' \ \mathcal{X}_{\mathcal{E};r} \ Q'$.
2. By Corollary B.23, whenever $P \xrightarrow{\nu\widetilde{b}.\overline{a}\langle M\rangle} P'$ with fresh $\widetilde{b}$ and $a \in r$, we have a Q' such that $Q \overset{\nu\widetilde{c}.\overline{a}\langle N\rangle}{\Longrightarrow} Q'$ with fresh $\widetilde{c}$ and $P' \ \mathcal{Y}^-_{(M,N)\oplus\mathcal{E};r} \ Q'$, i.e. $P' \ \mathcal{X}_{(M,N)\oplus\mathcal{E};r} \ Q'$.
3. By Corollary B.25, whenever $P \xrightarrow{a(M)} P'$ with $a \in r$, we have for all $(M, N) \in (\mathcal{E}; r)^\star$ a Q' such that $Q \overset{a(N)}{\Longrightarrow} Q'$ with $P' \ \mathcal{Y}^-_{\mathcal{E};r} \ Q'$, i.e. $P' \ \mathcal{X}_{\mathcal{E};r} \ Q'$.
4. By Lemma B.17, we have $P^+ \mid l[P_1^+] \ \mathcal{Y}^\star_{\mathcal{E}^+;r} \ Q^+ \mid l[Q_1^+]$ for some $P^+ \ \mathcal{Y}^\star_{\mathcal{E}^+;r} \ Q^+$ with $P \leq P^+$, $Q \leq Q^+$, $\mathcal{E} \subseteq \leq \mathcal{E}^+ \geq$, and $(`P_1, `Q_1) \leq (`P_1^+, `Q_1^+) \in \leq \mathcal{E}^+ \geq$, whose existence is guaranteed by definition of $\mathcal{Y}^-$. Then, by *run*-erasure, we have $P \mid l[P_1] \ \mathcal{Y}^-_{\mathcal{E};r} \ Q \mid l[Q_1]$.
5. By Lemma B.16, we have for any n not in $fn(\mathcal{E}, P, Q)$, $P \ \mathcal{Y}^-_{\mathcal{E};n\oplus r} \ Q$, i.e. $P \ \mathcal{X}_{\mathcal{E};n\oplus r} \ Q$.
6. Similarly, the converse of the first three clauses holds too.

Theorem B.31. [Originally Theorem 1, "Barbed equivalence from environmental bisimulation"]
If $P \ \mathcal{Y}^-_{\emptyset;fn(P,Q)} \ Q$ for a simple environmental bisimulation up-to context $\mathcal{Y}$, then $P \approx Q$.

Proof. We know by Theorem B.30 that $\mathcal{Y}^-$ is an environmental bisimulation. We let $\mathcal{Z} = \{(P, Q) \mid P\, \mathcal{Y}^-_{\emptyset;fn(P,Q)}\, Q\}$ and prove that $\mathcal{Z}$ is included in $\approx$.

1. **Clause** $P \xrightarrow{\tau} P$
   As $\mathcal{Y}^-$ is an environmental bisimulation, by clause 1 of the bisimulation, there is Q' such that $Q \Rightarrow Q'$ and $P'\, \mathcal{Y}^-_{\emptyset;r}\, Q'$. Therefore, for $r' = fn(P', Q') \subseteq r$, we have $P'\, \mathcal{Y}^-_{\emptyset;r'}\, Q'$, hence $(P', Q') \in \mathcal{Z}$.
2. **Clause** $P \downarrow_\mu$
   There are two cases depending on μ:
   - **Case** $P \downarrow_a$
     We have that $P \xrightarrow{a(V)} P'$ for some $(V, W) \in (\emptyset; r)^\star$ and P'. Since $a \in r$, by $\mathcal{Y}^-$ being an environmental bisimulation and clause 2 of the bisimulation, there is also Q' such that $Q \xRightarrow{a(W)} Q'$, that is, $Q \Downarrow_a$.
   - **Case** $P \downarrow_{\overline{a}}$
     We have that $P \xrightarrow{\nu\widetilde{c}.\overline{a}\langle V\rangle} P'$ for some fresh $\widetilde{c}$, V, and P'. Since $a \in r$, by $\mathcal{Y}^-$ being an environmental bisimulation and clause 3 of the bisimulation, there are also W, Q' and fresh $\widetilde{d}$ such that $Q \xRightarrow{\nu\widetilde{d}.\overline{a}\langle W\rangle} Q'$, that is, $Q \Downarrow_a$.
3. **Clause** Converse of 1, 2 on Q
   Similar to 1, 2.
4. **Clause** R a process
   Let $r' = fn(R)$; by appealing to the clause 5 of the bisimulation, since the names in r' are either fresh or already in r, we have that $P\, \mathcal{Y}^-_{\emptyset;r\cup r'}\, Q$. Also, $R\, (\emptyset; r \cup r')^\circ\, R$ and thus, using the up-to context technique, $P \mid R\, \mathcal{Y}^-_{\emptyset;r\cup r'}\, Q \mid R$ since $\mathcal{Y}^-$ is preserved by parallel composition of processes from $(\emptyset; r \cup r')^\circ$. Therefore, $(P \mid R, Q \mid R) \in \mathcal{Z}$.

3 Reduction-closed barbed congruence from environmental bisimulations

Definition B.32. [Size of a simple process]
We define inductively the size $size(P)$ of a simple process P as

$$size(P) = \begin{cases} 0 & \textit{if } P = 0 \\ 1 + size(Q) & \textit{if } P = a.Q \\ 1 + size(Q) + size(R) & \textit{if } P = \overline{a}\langle Q\rangle.R \\ max(size(Q), size(R)) & \textit{if } P = Q|R \\ 1 + size(Q) & \textit{if } P = l[Q] \\ size(Q) & \textit{if } P = !Q \\ size(Q) & \textit{if } P = run`Q \end{cases}$$

Notice that the size is impervious to the run constructor, and that therefore $size(P) = size(P^+)$ for all $P \leq P^+$. Also, $size(P) = 0$ if and only if $fn(P) = \emptyset$.

Lemma B.33. [Size and transitions]
For all simple process P, for all P', α, if $P \xrightarrow{\alpha} P'$ then

– $fn(P) \supseteq fn(P')$.
– $size(P) \geq size(P')$. *Moreover, if* $\alpha = \overline{a}\langle `P''\rangle$, $size(P) > size(P'')$ *holds too.*

Proof. By induction on the transition derivations of $P \xrightarrow{\alpha} P'$.

Lemma B.34. [Names and context closure]
If for any $(M, N) \in \mathcal{E}$, $fn(M) = fn(N)$ *then for any* r *and any* $(M_c, N_c) \in (\mathcal{E}; r)^\star$, $fn(M_c) = fn(N_c)$.

Proof. By structural induction on M_c's context.

Lemma B.35. [Free names and simple environmental bisimulation up-to context]
Let $\mathcal{Y}$ *be a simple environmental bisimulation up-to context. If* $0\, \mathcal{Y}_{\mathcal{E};fn(\mathcal{E})}\, 0$, *then for all* $(`P, `Q) \in \mathcal{E}$, $fn(P) = fn(Q)$.

Proof. We show by induction on the size of P the contraposition: for all P, Q, $\mathcal{E}$ and $r = fn(\mathcal{E})$, if $(P, Q) \in \mathcal{E}$, then it holds that $fn(P) \neq fn(Q)$ implies $(r, \mathcal{E}, 0, 0) \notin \mathcal{Y}$.

– Case $size(P) = 0$
We know by its size that P has no free name. However, by $fn(P) \neq fn(Q)$, there is a name $a \in fn(Q)$ such that there is a weak transition $Q \overset{run}{\Longrightarrow} \xrightarrow{\alpha} Q'$ with $a \in fn(\alpha)$. Thus, if $(r, \mathcal{E}, 0, 0) \in \mathcal{Y}$ holds, we have $run(`P)\, \mathcal{Y}^\star_{\mathcal{E};r}\, run(`Q)$ by up-to context, and by $Q \overset{run}{\Longrightarrow} \xrightarrow{\alpha}$ (hence $run(`Q) \overset{run}{\Longrightarrow} \xrightarrow{\alpha}$) and Lemma B.22 or Lemma B.24 we should have $run(`P) \overset{\alpha}{\Rightarrow}$. However, by Lemma B.33, P will never weakly exhibit α and we thus reached a contradiction, hence $(r, \mathcal{E}, 0, 0) \notin \mathcal{Y}$.
– Case $size(P) > 0$
Assuming $fn(P) > fn(Q)$, we know there is a name $a \in fn(P) \setminus fn(Q)$ such that one of the two following cases holds.
  1. $P \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} \xrightarrow{\overline{o}\langle M\rangle}$ with $a \in fn(M)$.
  Suppose $0\, \mathcal{Y}_{\mathcal{E};r}\, 0$, we have $l[P]\, \mathcal{Y}^\star_{\mathcal{E};l\oplus r}\, l[Q]$ for some l not in r by clauses 5 and 4 of environmental bisimulation up-to context. Then, we have $l[P] \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} \xrightarrow{\overline{o}\langle M\rangle} l[P']$, hence $l[Q] \overset{\alpha_1}{\Rightarrow} \ldots \overset{\alpha_n}{\Rightarrow} \overset{\overline{o}\langle N\rangle}{\Longrightarrow} l[Q']$, with $a \notin fn(N)$ by Lemma B.35 and $l[P']\, \mathcal{Y}^-_{(M,N)\oplus\ldots\oplus\mathcal{E};l\oplus r}\, l[Q']$. By Lemma B.33 we know that $size(M) < size(P)$. By Corollary B.23, we have $0\, \mathcal{Y}^-_{(`P',`Q')\oplus(M,N)\oplus\ldots\oplus\mathcal{E};r}\, 0$. Therefore, we also have an $\mathcal{E}'$ such that $(`P'^+, `Q'^+)\oplus(M^+, N^+)\oplus\ldots\oplus\mathcal{E} \subseteq (\mathcal{E}'; r)^\star$ with $(M, N) \leq (M^+, N^+)$. There is therefore a pair of terms $(`P_m, `Q_m) \in \mathcal{E}'$ such that $(M^+, N^+) = (C[`\widetilde{A}, `P_m], C[`\widetilde{B}, `Q_n])$ and $fn(P_m) \neq fn(Q_m)$. Naturally, by $size(M^+) = size(M) < size(P)$, we have $size(P_m) < size(P)$. We thus apply the induction hypothesis to P_m and $\mathcal{E}'$, and obtain that $(r, \mathcal{E}', 0, 0) \notin \mathcal{Y}$. Therefore, $(r, (`P'^+, `Q'^+)\oplus(M^+, N^+)\oplus\ldots\oplus\mathcal{E}, 0, 0) \notin \mathcal{Y}^\star$, hence $(r, (`P', `Q')\oplus(M, N)\oplus\ldots\oplus\mathcal{E}, 0, 0) \notin \mathcal{Y}^-$. However, $(r, (`P', `Q')\oplus(M, N)\oplus \ldots\oplus\mathcal{E}, 0, 0) \in \mathcal{Y}^-$ was a requirement from $(r, \mathcal{E}, 0, 0) \in \mathcal{Y}$ which therefore does not hold.
  2. Either $P \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} \xrightarrow{\overline{a}\langle\cdot\rangle}$ or $P \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} \xrightarrow{a}$.
  Assuming $0\, \mathcal{Y}_{\mathcal{E};r}\, 0$, we have $l[P]\, \mathcal{Y}^\star_{\mathcal{E};r}\, l[Q]$ and $P \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} P' \xrightarrow{\alpha}$ with $\alpha = \overline{a}\langle\cdot\rangle$ or $\alpha = a$, hence $Q \overset{\alpha_1}{\Rightarrow} \ldots \overset{\alpha_n}{\Rightarrow} Q'$, but, by Lemma B.33, we do not have $Q' \overset{\alpha}{\Rightarrow}$, which contradicts $0\, \mathcal{Y}_{\mathcal{E};r}\, 0$.

The situation $fn(P) < fn(Q)$ is handled similarly.

Lemma B.36. [Barbed congruence from *run*-erasure and simple environmental bisimulation up-to context]
Let $\mathcal{Y}$ be the simple environmental bisimilarity up-to context, and $\mathcal{S} = \{(P,Q) \mid P \le C[\widetilde{P}^+],\ Q \le C[\widetilde{Q}^+],\ \exists\mathcal{E}.(\widetilde{P}^+, \widetilde{Q}^+) \in \mathcal{E},\ 0\,\mathcal{Y}_{\mathcal{E};fn(\widetilde{P}^+,\widetilde{Q}^+)}\,0\}$, for contexts C for processes. We show that for all closed $(P,Q) \in \mathcal{S}$, if $P \downarrow_\mu$ then $Q \Downarrow_\mu$, and that if $P \to P'$ then $Q \Rightarrow Q'$ for some Q' with $(P',Q') \in \mathcal{S}$, and conversely.

Proof. By induction on the transition derivation of $P \xrightarrow{\alpha} P'$ with $(P,Q) \in \mathcal{S}$. We prove the two properties separately. In both situations, there is a case analysis on who does the transition: the context's erasure or some P_i. By symmetricity, we do not show the converse proofs on Q's transition; they are similar. We write *run**‘(P) to mean *run*‘$(\ldots(\textit{run}‘(P))\ldots)$, and C_p^-, C_q^- for two possibly different erasures of the same C.

Barbs: the cases necessary to check for barbs are HO-IN, HO-OUT, PAR-R, PAR-L, REP, EXTR, GUARD, TRANSP, and PASSIV.

- HO-IN
  - **Subcase** $C_p^-[\widetilde{P}] = a(X).C_{1p}^-[\widetilde{P}]$
    If $C_p^-[\widetilde{P}] \downarrow_a$, we have $a(X).C_{1p}^-[\widetilde{P}] \xrightarrow{a(M)} \cdot$. Thus $C_q^-[\widetilde{Q}] = \textit{run}^*‘(a(X).C_{1q}^-[\widetilde{Q}]) \xRightarrow{run} a(X).C_{1q}^-[\widetilde{Q}]) \xrightarrow{a(N)} \cdot$, i.e. $C_q^-[\widetilde{Q}] \Downarrow_a$
  - **Subcase** $[\,]_i$
    By $(r,\mathcal{E},0,0) \in \mathcal{Y}$, we have $P_i\ \mathcal{Y}^-_{\mathcal{E};r}\ Q_i$, hence $Q_i \Downarrow_a$ if $P_i \downarrow_a$, hence *run**‘$Q_i \Downarrow_a$, that is $C_p^-[\widetilde{Q}] \Downarrow_a$.
- HO-OUT
  - **Subcase** $C_p^-[\widetilde{P}] = \overline{a}\langle D_p^-[\widetilde{P}]\rangle.C_{1p}^-[\widetilde{P}]$
    If $C_p^-[\widetilde{P}] \downarrow_{\overline{a}}$, we have $\overline{a}\langle‘D_p^-[\widetilde{P}]\rangle.C_{1p}^-[\widetilde{P}] \xrightarrow{\overline{a}\langle‘D_p^-[\widetilde{P}]\rangle} \cdot$. Thus, $C_q^-[\widetilde{Q}] = \textit{run}^*‘(\overline{a}\langle‘D_p^-[\widetilde{Q}]\rangle.C_{1q}^-[\widetilde{Q}]) \xRightarrow{run} \overline{a}\langle‘D_p^-[\widetilde{Q}]\rangle.C_{1q}^-[\widetilde{Q}]) \xrightarrow{\overline{a}\langle‘D_p^-[\widetilde{Q}]\rangle} \cdot$ i.e. $C_q^-[\widetilde{Q}] \Downarrow_{\overline{a}}$
  - **Subcase** $[\,]_i$
    By $(r,\mathcal{E},0,0) \in \mathcal{Y}$, we have $P_i\ \mathcal{Y}^-_{\mathcal{E};r}\ Q_i$, hence $Q_i \Downarrow_{\overline{a}}$ if $P_i \downarrow_{\overline{a}}$, hence *run**‘$Q_i \Downarrow_{\overline{a}}$, that is $C_p^-[\widetilde{Q}] \Downarrow_{\overline{a}}$.
    In all the other cases, this subcase is similar, and we will thus not write it anymore.
- PASSIV
  $C_p^-[\widetilde{P}] = a[C_{1p}^-[\widetilde{P}]] \downarrow_{\overline{a}}$. Trivially $a[C_{1q}^-[\widetilde{Q}]] \downarrow_{\overline{a}}$, hence $C_q^-[\widetilde{Q}] = \textit{run}^*‘(a[C_{1q}^-[\widetilde{Q}]]) \Downarrow_{\overline{a}}$.
- PAR-L
  $C_p^-[\widetilde{P}] = C_{1p}^-[\widetilde{P}] \mid C_{2p}^-[\widetilde{P}] \downarrow_\mu$ hence $C_{1p}^-[\widetilde{P}] \downarrow_\mu$. By the induction hypothesis, $C_{1q}^-[\widetilde{Q}] \Downarrow_\mu$ hence $C_q^-[\widetilde{Q}] = \textit{run}^*‘(C_{1q}^-[\widetilde{Q}] \mid C_{2q}^-[\widetilde{Q}]) \Downarrow_\mu$.
- PAR-R
  Similarly.
- REP
  $C_p^-[\widetilde{P}] = !C_{1p}^-[\widetilde{P}] \downarrow_\mu$, hence $!C_{1p}^-[\widetilde{P}] \mid C_{1p}^-[\widetilde{P}] \downarrow_\mu$. By the induction hypothesis, $!C_{1q}^-[\widetilde{Q}] \mid C_{1q}^-[\widetilde{Q}] \Downarrow_\mu$, hence $C_q^-[\widetilde{Q}] = \textit{run}^*‘(!C_{1q}^-[\widetilde{Q}]) \Downarrow_\mu$.

– EXTR
$C_p^-[\widetilde{P}] = \nu x.C_{1p}^-[\widetilde{P}] \downarrow_\mu$, hence $C_{1p}^-[\widetilde{P}] \downarrow_\mu$. By the induction hypothesis, $C_{1q}^-[\widetilde{Q}] \Downarrow_\mu$, hence $\nu x.C_{1q}^-[\widetilde{Q}] \Downarrow_\mu$ since the same barb $\mu \neq x, \overline{x}$ is used, hence $C_q^-[\widetilde{Q}] = \mathit{run}^*`(\nu x.C_{1q}^-[\widetilde{Q}]) \Downarrow_\mu$.

– GUARD
Similarly.

– TRANSP
$C_p^-[\widetilde{P}] = a[C_{1p}^-[\widetilde{P}]] \downarrow_\mu$, hence $C_{1p}^-[\widetilde{P}] \downarrow_\mu$. By the induction hypothesis, $C_{1q}^-[\widetilde{Q}] \Downarrow_\mu$, hence $a[C_{1q}^-[\widetilde{Q}]] \Downarrow_\mu$, hence $C_q^-[\widetilde{Q}] = \mathit{run}^*`(a[C_{1q}^-[\widetilde{Q}]]) \Downarrow_\mu$.

Reductions: the cases necessary to check for reduction closure are RUN, TRANSP, PAR-L, PAR-R, REP, GUARD, REACT-L, and REACT-R.

– RUN
- $run(`C_{1p}^-[\widetilde{P}]) \to C_{1p}^-[\widetilde{P}]$
  We have $run(`C_{1p}^-[\widetilde{P}]) \to C_{1p}^-[\widetilde{P}]$ and $C_{1p}^-[\widetilde{P}] \leq C[\widetilde{P}^+]$. We still have $C_q^-[\widetilde{Q}] \leq C[\widetilde{Q}^+]$, so $(P', Q) \in \mathcal{S}$ and we are done.
- $[\,]_i$
  We have $a[P_i] \, \mathcal{Y}_{\mathcal{E};r}^- \, a[Q_i]$ and $a[P_i] \to a[P_i'] \xrightarrow{\overline{a}\langle `P_i' \rangle} 0$, so $a[Q_i] \Rightarrow a[Q_i'] \xRightarrow{\overline{a}\langle `Q_i \rangle} 0$, and $0 \, \mathcal{Y}_{(`P_i', `Q_i') \oplus \mathcal{E};r}^- \, 0$, hence $(P_i', Q_i') \in \mathcal{S}$. Also, $\mathit{run}^*(`Q_i) \Rightarrow Q_i \Rightarrow Q_i'$ and we are done.

– TRANSP
- $a[C_{1p}^-[\widetilde{P}]] \to a[R]$
  We have $C_{1p}^-[\widetilde{P}] \to R$, so, by the induction hypothesis $C_{1q}^-[\widetilde{Q}] \Rightarrow S$ and $(R, S) \in \mathcal{S}$. Therefore $a[C_{1q}^-[\widetilde{Q}]] \Rightarrow a[S]$, and $(a[R], a[S]) \in \mathcal{S}$
- $[\,]_i$
  We have $a[P_i] \, \mathcal{Y}_{\mathcal{E};r}^- \, a[Q_i] \; a[P_i] \to a[P_i'] \xrightarrow{\overline{a}\langle `P_i' \rangle} 0$. So, $a[Q_i] \Rightarrow a[Q_i'] \xRightarrow{\overline{a}\langle `Q_i' \rangle} 0$, and $0 \, \mathcal{Y}_{(`P_i', `Q_i') \oplus \mathcal{E};r}^- \, 0$, hence $(a[P_i'], a[Q_i']) \in \mathcal{S}$. Also, $\mathit{run}^*(`Q_i) \Rightarrow Q_i \Rightarrow Q_i'$ and we are done. Again, this subcase always holds similarly, and thus it is not repeated below.

– PAR-L
$C_p^-[\widetilde{P}] = C_{1p}^-[\widetilde{P}] \mid C_{2p}^-[\widetilde{P}] \to R \mid C_{2p}^-[\widetilde{P}]$, i.e. $C_{1p}^-[\widetilde{P}] \to R$. So, by the induction hypothesis, $C_{1q}^-[\widetilde{Q}] \Rightarrow S$ and $(R, S) \in \mathcal{S}$, hence $C_q^-[\widetilde{Q}] = \mathit{run}^*`(C_{1q}^-[\widetilde{Q}] | C_{2q}^-[\widetilde{Q}]) \Rightarrow S \mid C_{2q}^-[\widetilde{Q}]$ and $(R \mid C_{2p}^-[\widetilde{P}], S \mid C_{2q}^-[\widetilde{Q}]) \in \mathcal{S}$.

– PAR-R
Similarly

– REP
$C_p^-[\widetilde{P}] = !C_{1p}^-[\widetilde{P}] \to R$, hence $!C_{1p}^-[\widetilde{P}] \mid C_{1p}^-[\widetilde{P}] \to R$. By the induction hypothesis, $!C_{1q}^-[\widetilde{Q}] \mid C_{1q}^-[\widetilde{Q}] \Rightarrow S$ with $(R, S) \in \mathcal{S}$, hence, $!C_{1q}^-[\widetilde{Q}] \Rightarrow S$, $C_q^-[\widetilde{Q}] = \mathit{run}^*`(!C_{1q}^-[\widetilde{Q}]) \Rightarrow S$, and still $(R, S) \in \mathcal{S}$.

– GUARD
$C_p^-[\widetilde{P}] = \nu x.C_{1p}^-[\widetilde{P}] \to \nu x.R$, i.e. $C_{1p}^-[\widetilde{P}] \to R$. So, by the induction hypothesis,

$C^{-}_{1q}[\widetilde{Q}]{\Rightarrow}S$ and $(R,S)\in\mathcal{S}$ hence $vx.C^{-}_{1q}[\widetilde{Q}]{\Rightarrow}vx.S$, $C^{-}_{q}[\widetilde{Q}] = \mathit{run}^{*}`(\nu x.C^{-}_{1q}[\widetilde{Q}]){\Rightarrow}$ $\nu x.S$ and $(\nu x.R, \nu x.S)\in\mathcal{S}$.

- REACT-L
  There are several subcases.
  - The two contexts react
    $C^{-}_{p}[\widetilde{P}] = C^{-}_{1p}[\widetilde{P}_0,\widetilde{P}_1] \mid C^{-}_{2p}[\widetilde{P}_2] \rightarrow \nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_0] \mid C'^{-}_{2p}[\widetilde{P}_2,\widetilde{P}_1])$. Of course, C^{-}_{1q}can (weakly) do the same reaction, and since $fn(\widetilde{P}_1) = fn(\widetilde{Q}_1)$ by Lemma B.35, the same names $\widetilde{x}$ are extruded, giving $C^{-}_{q}[\widetilde{Q}] \Rightarrow \nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_0] \mid C'^{-}_{2q}[\widetilde{Q}_2,\widetilde{Q}_1])$ and as expected $(\nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_0] \mid C'^{-}_{2p}[\widetilde{P}_2,\widetilde{P}_1]), \nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_0] \mid C'^{-}_{2q}[\widetilde{Q}_2,\widetilde{Q}_1])) \in \mathcal{S}$.
  - $C^{-}_{1p}[\widetilde{P}_1]$ sends, P_i in $C^{-}_{2p}[\widetilde{P}_2, P_i]$ receives.
    $C^{-}_{p}[\widetilde{P}] = C^{-}_{1p}[\widetilde{P}_1] \mid C^{-}_{2p}[\widetilde{P}_2, P_i] \rightarrow \nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_1] \mid C'^{-}_{2p}[\widetilde{P}_2, P'_i])$. We know that $C^{-}_{1q}[\widetilde{Q}]$ can weakly do the same output transition, that it will extrude the same names. Also, we have $a[P_i]\ \mathcal{Y}^{-}_{\mathcal{E};r}\ a[Q_i]$ and $a[P_i] \xrightarrow{o} a[P'_i] \xrightarrow{\overline{a}\langle`P'_i\rangle} 0$ for some channel $o\in r$. So, $a[Q_i]\overset{o}{\Rightarrow}a[Q'_i]\xrightarrow{\overline{a}\langle`Q'_i\rangle}0$ with $(r, (`P'_i, `Q'_i)\oplus\mathcal{E}, 0, 0) \in \mathcal{Y}^{-}$, so $C^{-}_{q}[\widetilde{Q}]{\Rightarrow}\nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_1]|C'^{-}_{2q}[\widetilde{Q}_2, Q'_i])$, and $(\nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_1]|C'^{-}_{2p}[\widetilde{P}_2, P'_i]), \nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_1]| C'^{-}_{2q}[\widetilde{Q}_2, Q'_i])) \in \mathcal{S}$.
  - $C^{-}_{2p}[\widetilde{P}_2]$ receives, P_i in $C^{-}_{1p}[\widetilde{P}_1, P_i]$ sends
    $C^{-}_{p}[\widetilde{P}] = C^{-}_{1p}[\widetilde{P}_1, P_i]|C^{-}_{2p}[\widetilde{P}_2]{\rightarrow}\nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_1, P'_i]|C^{-}_{2p}[\widetilde{P}_2, P_j])$. We know that $C^{-}_{2q}[\widetilde{Q}_2]$ can weakly do the same input transition. Also, we have $a[P_i]\ \mathcal{Y}^{-}_{\mathcal{E};r}\ a[Q_i]$ and $a[P_i] \xrightarrow{\overline{o}\langle`P_j\rangle} a[P'_i] \xrightarrow{\overline{a}\langle`P'_i\rangle} 0$, so, $a[Q_i] \overset{\overline{o}\langle`Q_j\rangle}{\Longrightarrow} a[Q'_i] \xrightarrow{\overline{a}\langle`Q'_i\rangle} 0$ with $(r, (`P_j, `Q_j)\oplus(`P'_i, `Q'_i)\oplus\mathcal{E}, 0, 0) \in \mathcal{Y}^{-}$ and the same names in (P_j, Q_j) by Lemma B.35, so $C^{-}_{q}[\widetilde{Q}]{\Rightarrow}\nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_1, Q'_i]|C'^{-}_{2q}[\widetilde{Q}_2, Q_j])$, and $(\nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_1, P'_i]| C'^{-}_{2p}[\widetilde{P}_2, P_j]), \nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_1, Q'_i] \mid C'^{-}_{2q}[\widetilde{Q}_2, Q_j])) \in \mathcal{S}$.
  - P_i in $C^{-}_{1p}[\widetilde{P}_1, P_i]$ and P_j in $C^{-}_{2p}[\widetilde{P}_2, P_j]$ react
    $C^{-}_{p}[\widetilde{P}] = C^{-}_{1p}[\widetilde{P}_1, P_i] \mid C^{-}_{2p}[\widetilde{P}_2, P_j] {\rightarrow} \nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_1, P'_i] \mid C^{-}_{2p}[\widetilde{P}_2, P'_j])$. We have $a[P_i] \mid b[P_j]\ \mathcal{Y}^{-}_{\mathcal{E};r}\ a[Q_i] \mid b[Q_j]$ and $a[P_i] \mid b[P_j] \rightarrow a[P'_i] \mid b[P'_j] \xrightarrow{\overline{a}\langle`P'_i\rangle}\xrightarrow{\overline{b}\langle`P'_j\rangle}0$, so $a[Q_i]|b[Q_j]{\Rightarrow}a[Q'_i]|b[Q'_j]\xrightarrow{\overline{a}\langle`Q'_i\rangle}\xrightarrow{\overline{b}\langle`Q'_j\rangle}0$ with $(r, (`P'_j, `Q'_j)\oplus(`P'_i, `Q'_i)\oplus\mathcal{E}, 0, 0) \in \mathcal{Y}^{-}$ and the same names communicated from P_i, Q_i to P_j, Q_j by Lemma B.35, so $C^{-}_{q}[\widetilde{Q}]{\Rightarrow}\nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_1, Q'_i]|C'^{-}_{2q}[\widetilde{Q}_2, Q'_j])$, and $(\nu\widetilde{x}.(C'^{-}_{1p}[\widetilde{P}_1, P'_i]|C'^{-}_{2p}[\widetilde{P}_2, P'_j])$, $\nu\widetilde{x}.(C'^{-}_{1q}[\widetilde{Q}_1, Q'_i] \mid C'^{-}_{2q}[\widetilde{Q}_2, Q'_j])) \in \mathcal{S}$
- REACT-R
  Similarly

Corollary B.37. [Originally Corollary 1, "Barbed congruence from environmental bisimulation"]
If $\overline{a}\langle`P\rangle\ \mathcal{Y}^{-}_{\emptyset;a\oplus fn(P,Q)}\ \overline{a}\langle`Q\rangle$ for a simple environmental bisimulation up-to context $\mathcal{Y}$, then $P \approx_c Q$.

Proof. By $\overline{a}\langle`P\rangle \; \mathcal{Y}_{\emptyset;a\oplus fn(P,Q)} \; \overline{a}\langle`Q\rangle$, we have $0 \; \mathcal{Y}^{\star}_{(`P,`Q);a\oplus fn(P,Q)} \; 0$, hence $0 \; \mathcal{Y}_{\mathcal{E};r} \; 0$ for an environment $\mathcal{E}$ and names r such that $(`P,`Q) \in (\mathcal{E};r)^{\star}$ and $a\oplus fn(`P,`Q) \subseteq r$. By Lemma B.36, we have $(P,Q) \in \mathcal{S}$, for some $\mathcal{S}$ included in $\approx_c$.

C Examples

We write $P \mid \ldots \mid P$ for a finite, possibly null, product of the process P.

Example C.1. [Originally Example 2]
$!\overline{a} \mid !e \; \approx \; !a[e]$.

Proof. Take $\mathcal{X} = \{(r,\ \mathcal{E},\ P,\ Q) \mid r \supseteq \{a, e, l_1, \ldots, l_n\} \mid \mathcal{E} = \{(`0,`e)\},\ n \geq 0,\ P = !\overline{a} \mid !e \mid \prod_{i=1}^{n} l_i[0],\ Q = !a[e] \mid \prod_{i=1}^{n} l_i[e] \mid a[0] \mid \ldots \mid a[0]\}$. As expected, the definition of $\mathcal{X}$ relates the original processes $!\overline{a}|!e$ and $!a[e]$. Clause 4 of the bisimulation requires that for any $(`P_1,`Q_1) \in \mathcal{E}$ and $l \in r$, we have $P_0 \mid l[P_1] \; \mathcal{X}_{\mathcal{E};r} \; Q_0 \mid l[Q_1]$. Repeatedly applying this clause explains the presence of the products $\prod_{i=1}^{n} l_i[\cdot]$ in both P and Q. All outputs that P and Q can do are respectively $\{`0\}$ and $\{`0,`e\}$. However, since $(`0,`0) \in (\{(`0,`e)\};\emptyset)^{\star}$, we can work up-to context with smaller environment $\mathcal{E} = \{(`0,`e)\}$. Finally, at least the free names $a, e, l_1, \ldots, l_n$ of P and Q and maybe more fresh names are in r, so to satisfy clause 5 of the bisimulation and to guarantee behavioural equivalence.

Let us now check the output clause of the bisimulation, starting with passivation of $l_i[\cdot]$ which is trivial to verify since all output terms are already related by $\mathcal{E}$. Otherwise, P can only output by doing $P \xrightarrow{\overline{a}\langle`0\rangle} \equiv P$, and $Q$ can follow with $Q \xrightarrow{\overline{a}\langle`e\rangle} \equiv Q$ with still $(r, \mathcal{E}, P, Q) \in \mathcal{X}$. Conversely, when Q outputs to a either $`0$ or $`e$, P follows with an action $\overline{a}\langle`0\rangle$ from a replication of $\overline{a}$, and we are done.

As far as inputs are concerned, whenever $P \xrightarrow{e} \equiv P$, Q can consume the e in a copy of $a[e]$ to become a process $Q|a[0]$ (hence the list $a[0]|\ldots|a[0]$), which is still related by $\mathcal{X}$ to P under r and $\mathcal{E}$. The converse transition is treated similarly. Q can also input in $l_i[e]$ with a transition $l_i[e] \xrightarrow{e} l_i[0]$, in which case P can input on a replication from $!e$, doing $P \xrightarrow{e} \equiv P$. We can remove up-to context the $l_n[0]$ of Q's transition and P's original $l_n[0]$, and we are done since $(r, \mathcal{E}, !\overline{a} \mid !e \mid \prod_{i=1}^{n-1} l_i[0], !a[e] \mid \prod_{i=1}^{n-1} l_i[e] \mid a[0] \mid \ldots \mid a[0]) \in \mathcal{X}$.

Without loss of generality, internal transitions involve passivation of $l_n[\cdot]$ (necessarily with $l_n = e$), and are easily matched: (i) when $P = !\overline{a} \mid !e \mid \prod_{i=1}^{n} l_i[0] \xrightarrow{\tau} \equiv !\overline{a} \mid !e \mid \prod_{i=1}^{n-1} l_i[0]$, Q can do $!a[e] \mid \prod_{i=1}^{n} l_i[e] \mid a[0] \mid \ldots \mid a[0] \xrightarrow{\tau} \equiv !a[e] \mid \prod_{i=1}^{n-1} l_i[e] \mid a[0] \mid a[0] \mid \ldots \mid a[0]$ (which increases the count of $a[0]$'s); (ii) the converse of (i) by Q is similar, and (iii) when $Q = !a[e] \mid \prod_{i=1}^{n-2} l_i[e] \mid l_{n-1}[e] \mid l_n[e] \xrightarrow{\tau} \equiv !a[e] \mid \prod_{i=1}^{n-2} l_i[e] \mid l_{n-1}[0]$, P can passivate $l_n[0]$ using a replication from $!e$. The resulting processes are still related by $\mathcal{X}$ with r and $\mathcal{E}$, and we can even remove the $l_{n-1}[0]$'s up-to context.

We therefore have $!\overline{a} \mid !e \; \approx \; !a[e]$ from the soundness of environmental bisimulation up-to context.

Example C.2. [Originally Example 3]
$!a[e] \mid !b[\overline{e}] \approx !a[b[e|\overline{e}]]$. This example shows the equivalence proof of more complicated processes with nested locations.

Proof. Take:

$$\begin{aligned}
\mathcal{X} = \{(r,\ \mathcal{E},\ P,\ Q) \mid r \ &\supseteq \{a, e, b, l_1, \ldots, l_n\},\\
P_0 &= !a[e] \mid !b[\overline{e}],\ Q_0 = !a[b[e \mid \overline{e}]],\\
P \ &= P_0 \mid \textstyle\prod_{i=1}^n l_i[P_i] \mid b[0] \mid \ldots \mid b[0],\\
Q \ &= Q_0 \mid \textstyle\prod_{i=1}^n l_i[Q_i],\\
(\widetilde{P}, \widetilde{Q}) &\in \mathcal{E},\ n \geq 0\},\\
\mathcal{E} \ = \{(`x, `y) \mid x \in \{0, e, \overline{e}\},\ y \equiv\in &\{0, e, \overline{e}, (e \mid \overline{e}), b[0], b[e], b[\overline{e}], b[e \mid \overline{e}]\}\}\,.
\end{aligned}$$

$\mathcal{X}$ relates considered processes P_0 and Q_0 together with a set r containing at least the free names of P and Q, and an environment $\mathcal{E}$ defined as the Cartesian product of all terms that we expect P and Q to output. As in Examples 1 and C.1, we could omit pairs of the form $(`x, `x)$ from $\mathcal{E}$, but this would only complicate the definition. As in Example C.1, we have the products $\prod_{i=1}^n l_i[\cdot]$ by clause 4 of the bisimulation. Process P may also contain a list of subprocesses $b[0] \mid \ldots \mid b[0]$ whose presence will be clarified below.

Let us now check the input transitions. When P inputs on e, it either uses a copy of $a[e]$, leaving behind $a[0]$, or uses P_n which becomes P'_n. Q_0 too can input on e, leaving a process $a[b[\overline{e}]]$. Then, in the former case, $a[0]$ and $a[b[\overline{e}]]$ can be merged with the products $\prod$ since $(`0, `b[\overline{e}]) \in \mathcal{E}$. Else, in the latter case, it holds that $(`P'_n, `Q_n) \in \mathcal{E}$; we can then draw a copy $a[e]$ from P_0 and, since $(`e, `b[\overline{e}]) \in \mathcal{E}$ too, we can merge Q's residue $a[b[\overline{e}]]$ and the copy $a[e]$ with their corresponding products $\prod$. Finally, inputs by Q are matched similarly. From now on, we shall refer as *pairing* to the drawing from P_0 or Q_0 of a process to pair with a residue so to enlarge the products $\prod$.

Outputs transitions are conceptually matched similarly: when $P_0 \xrightarrow{\overline{e}} P_0 \mid b[0]$, process Q can follow with $Q_0 \xrightarrow{\overline{e}}\equiv Q_0 \mid a[b[e]]$ (and conversely), but $b[0]$ and $a[b[e]]$ cannot be added together to the products $\prod$, for locations a and b differ. Yet, we can use pairing to handle $a[b[e]]$, and leave the residue $b[0]$ as is, therefore enlarging the list $b[0] \mid \ldots \mid b[0]$ in P. Outputs by P_n (resp. Q_n) are matched in a similar manner: output with a copy from Q_0 (resp. P_0) leaving a residue $a[\cdot]$, and pairing to handle this residue. Passivations of $l_i[\cdot]$ and $a[\cdot]$ are trivial to simulate, since $(`P_i, `Q_i)$ and $(`e, `b[e \mid \overline{e}])$ are in $\mathcal{E}$. Passivations of P's $b[\overline{e}]$ and $b[0]$ are matched by the passivation of $b[e \mid \overline{e}]$ in a replication drawn from Q_0, leaving a residue $a[0]$ paired as usual. Passivation of Q_0's $b[e \mid \overline{e}]$ is handled similarly. Finally, when $Q_n \xrightarrow{\overline{b}\langle `R\rangle} Q'_n = 0$ for $R \in \{0, e, \overline{e}, (e \mid \overline{e})\}$, we can just passivate a copy of $b[\overline{e}]$ from $P_0$ and we are done since $(`\overline{e}, `R)$ and $(`P_n, `0)$ are both in the environment $\mathcal{E}$.

We now check the τ transitions clause. If P_0 reduces to $P_0 \mid a[0] \mid b[0]$, or reacts with P_i leaving residues $a[0]$ or $b[0]$, we use pairing for the $a[0]$'s, and add the $b[0]$'s to $b[0] \mid \ldots \mid b[0]$. Conversely, Q_0 can reduce to $Q_0 \mid a[b[0]]$ or $Q_0 \mid a[b[e]] \mid a[b[\overline{e}]]$, or may react with Q_i, leaving residues $a[b[e]]$ or $a[b[\overline{e}]]$. All residues can be paired up with one or two copies of $a[e]$ from P_0, and in all the above reactions, integrity of the products is preserved (i.e. $(`\widetilde{P}, `\widetilde{Q}) \in \mathcal{E}$ still holds). Also, $l_n[\cdot]$ can be passivated (necessarily with $l_n = e$) in P by either $a[e]$ or P_{n-1}. Therefore P becomes either $P_0 \mid a[0] \mid \prod_{i=1}^{n-1} l_i[P_i] \mid b[0] \mid \ldots \mid b[0]$ or $P_0 \mid \prod_{i=1}^{n-2} l_i[P_i] \mid l_{n-1}[P'_{n-1}] \mid b[0] \mid \ldots \mid b[0]$. In both cases, Q follows with $Q \xrightarrow{\tau}\equiv Q_0 \mid a[b[\overline{e}]] \mid \prod_{i=1}^{n-1} l_i[Q_i]$, and either $a[0]$ and

$a[b[\overline{e}]]$ join the products $\prod$, or pairing is used to assure membership in $\mathcal{X}$. The converse for Q of those reactions with a passivation is similarly checked. Finally, for reactions involving only P_i's (resp. Q_i's), Q (resp. P) needs not do anything, for the integrity of the products $\prod$ is preserved.

This concludes our proof that the original processes $!a[e] \mid !b[\overline{e}]$ and $!a[b[e \mid \overline{e}]]$ are behaviourally equivalent.