Logical Relations for Encryption

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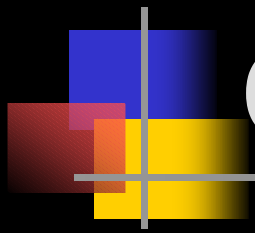
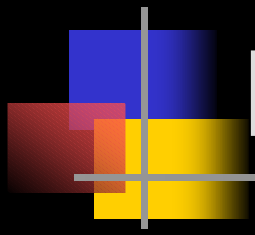
Joint work with Benjamin Pierce,
University of Pennsylvania



Overview

- Introduction
- The cryptographic λ-calculus
- Logical relations
- Application: protocol encoding
- Extensions
- Related work
- Conclusion

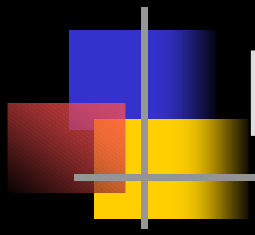


Motivation

Two approaches to information hiding:

- Encryption
  - mainly studied in security systems
- Type abstraction
  - mainly studied in programming languages (polymorphism, modules, objects, etc.)

How are these related?

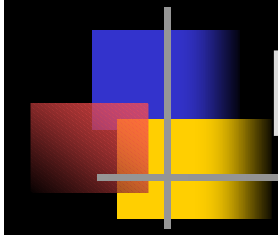


Results

Adapting the theory of type abstraction for encryption

- Cryptographic λ-calculus +
- Logical relation of the polymorphic λ-calculus

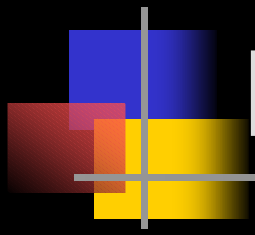
$\Rightarrow$ Method of proving secrecy in programs using encryption



Example

A program p(i) consisting of

- a secret integer i and
- an interface function $\lambda x.\ x \bmod 2$



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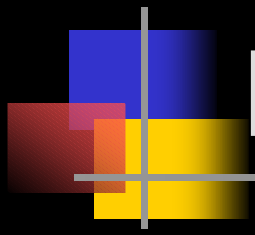

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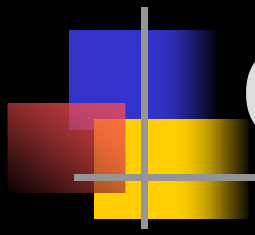
- Information hiding by encryption

$$p(i) = \text{new k in } \langle \{i\}_k, \lambda\{x\}_k.\ x \bmod 2 \rangle$$



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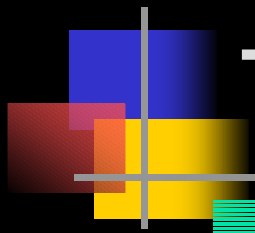
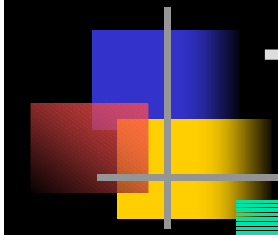


The Cryptographic λ-Calculus

Simply typed call-by-value λ-calculus + (perfect) cryptographic primitives

$e ::= \{e_1\}_{e2} \mid \text{let } \{x\}_{e1} = e_2 \text{ in } e_3 \text{ else } e_4$
$\mid \text{new } x \text{ in } e \mid k \mid \ldots$

$\tau ::= \text{bits}[\tau] \mid \text{key}[\tau] \mid \ldots$



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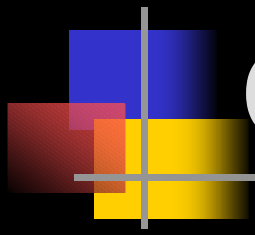
$\text{new } x \text{ in } e \rightarrow [k/x]e \quad (k \text{ fresh})$

$\text{let } \{x\}_{k1} = \{v\}_{k2} \text{ in } e_1 \text{ else } e_2$
$\rightarrow [v/x]e_1 \text{ (if } k_1 = k_2) \text{ or } e_2 \text{ (if } k_1 \neq k_2)$

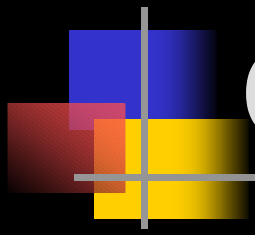
Secrecy ≅ Non-Interference ≅ Contextual Equivalence



[Q] How to state the (partial) secrecy of the value of i?

[A] By conditional non-interference:
if $i \equiv j \pmod 2$, then p(i) and p(j) are equivalent under any context

"Outsiders cannot observe the difference of the secret"



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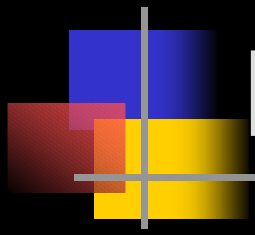


Logical Relation

[Q] How to prove contextual equivalence?

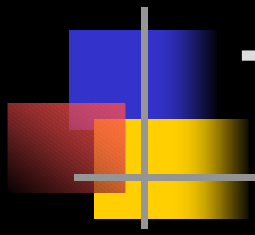
[A] By a logical relation "~" between programs, defined by induction on their type

Main theorem:

$$e_1 \sim e_2 : \tau \Rightarrow e_1 \approx e_2 : \tau$$

"related programs are contextually equivalent"

Logical Relation for Simple Types (standard)



- Integers are related iff they are equal

$$i \sim j : \text{int} \iff i = j$$

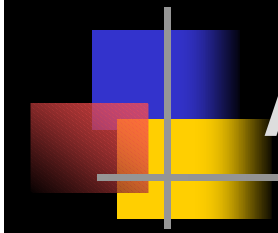
- Functions are related iff they return related results when applied to related arguments

$$f \sim g : \tau_1 \to \tau_2 \iff$$
$$f\ v \sim g\ w : \tau_2 \text{ for any } v \sim w : \tau_1$$

- Pairs are related iff their elements are related

$$(v_1, v_2) \sim (w_1, w_2) : \tau_1 \times \tau_2 \iff$$
$$v_1 \sim w_1 : \tau_1 \text{ and } v_2 \sim w_2 : \tau_2$$

Logical Relation for Type Abstraction (also standard)



The relation environment φ gives the relation $\varphi(\alpha)$ between values of each abstract type α

$\varphi \vdash v_1 \sim v_2 : \alpha \;\Leftrightarrow\; (v_1, v_2) \in \varphi(\alpha)$

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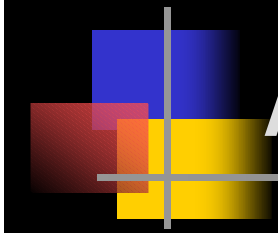

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$\varphi, \alpha \mapsto r \vdash e_1 \sim e_2 : \tau$ for some $r \subseteq \sigma_1 \times \sigma_2$

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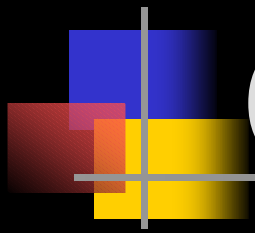
E.g., pack int, $\langle 1, \lambda x.\ x \bmod 2\rangle$ as $\exists\alpha.\alpha\times(\alpha\to\text{int})$
and pack int, $\langle 3, \lambda x.\ x \bmod 2\rangle$ as $\exists\alpha.\alpha\times(\alpha\to\text{int})$
can be related by taking $\alpha \mapsto \{(1,3)\}$

Logical Relation for Encryption (new!)



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$$\varphi \vdash \{v_1\}_{k1} \sim \{v_2\}_{k2} : \text{bits}[\tau] \iff (v_1, v_2) \in \varphi(k) \text{ where } k = k1 = k2$$

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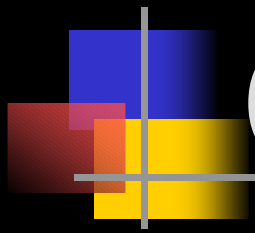

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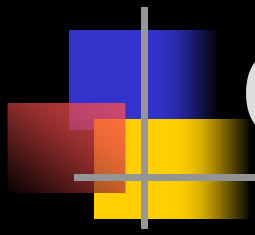

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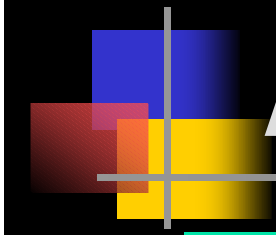

Application: Protocol Encoding

Encode:

- Sending of a message by the message itself
- Receiving of a message by a function
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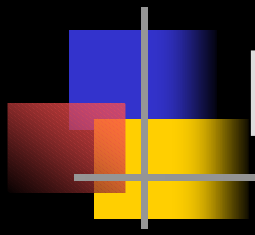
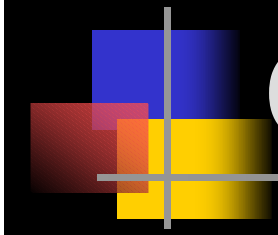
E.g., 1. $A \rightarrow B \quad \{i\}_k$
2. $B \rightarrow * \quad i \bmod 2$

- $p = \text{new } k \text{ in } \langle \{i\}_k, \lambda\{x\}_k.\ x \bmod 2 \rangle$
- $\text{Network}(p) = \#_2(p)\ \#_1(p) \rightarrow^* i \bmod 2$
- Attacker(p) = <u>any</u> context for p



Examples

- Well-known attack on (a bad use of) Needham-Schroeder public-key protocol
- Correctness proof of (the same use of) "improved" Needham-Schroeder public-key protocol
- "Necessarily parallel" attack on ffgg protocol



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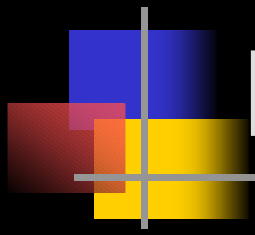


Extensions

- Recursive functions/types

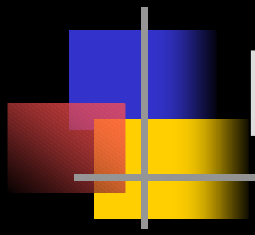
  for making the attackers Turing-complete

  - cf. [Pitts-98], [Crary-Harper], etc.

- State/linearity

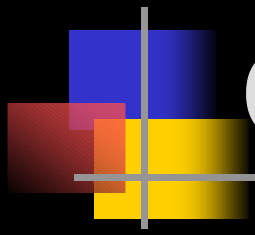
  for encoding protocols more precisely

  - cf. [Pitts-Stark-98], [Bierman-Pitts-Russo-00]



Related Work

- Logical relations
  - Relational parametricity [Reynolds-83]
  - Representation independence [Mitchell-91]
  - λ-calculus with name generation [Stark-94]
- Protocol verification
  - Various logics, theorem proving, model checking, etc. [many!]
  - In particular, spi-calculus [Abadi-Gordon]



Conclusion

- We have adapted the theory of type abstraction to encryption
- Can we do something in the other direction?

  E.g., implement type abstraction by encryption

  I.e., encode the polymorphic λ-calculus
  into the untyped cryptographic λ-calculus
  (while preserving contextual equivalence)

  $\Rightarrow$ Extend the scope of type abstraction from the statically typed world to the untyped world (such as open network)