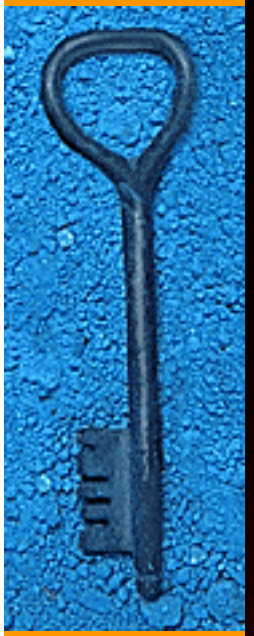
Relating Cryptography and Polymorphism

Eijiro Sumii
Joint Work with Benjamin Pierce
University of Pennsylvania

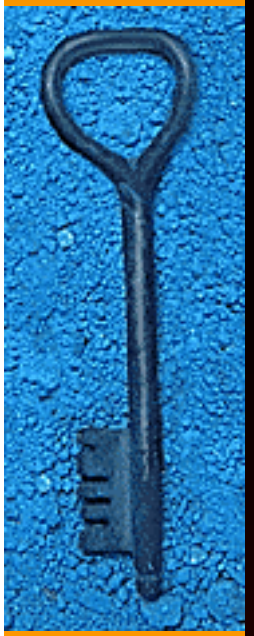


Main Result

- Adaptation of *relational parametricity* from polymorphic λ-calculus to *cryptographic λ-calculus*

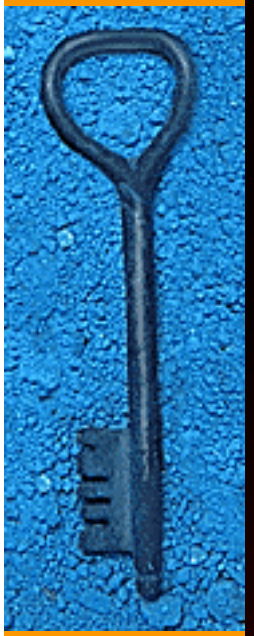
$$\vdash e \sim e' : \tau \;\Rightarrow\; e \approx e'$$

  - useful for reasoning about programs using encryption
    - E.g., (in)correctness proof of security protocols



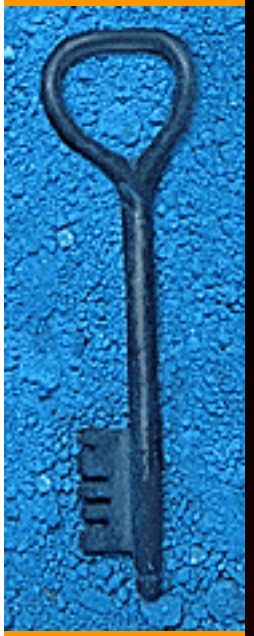
Outline

- Background
- Parametricity for Type Abstraction
- Cryptographic λ-Calculus
- Parametricity for Encryption
- Current Status and Future Work



Two Approaches to Information Hiding

- Type abstraction
  - conceals the types of data
    - existential types, universal types, modules, packages, etc.
- Encryption
  - obfuscates the values of data



Example

- Using type abstraction:

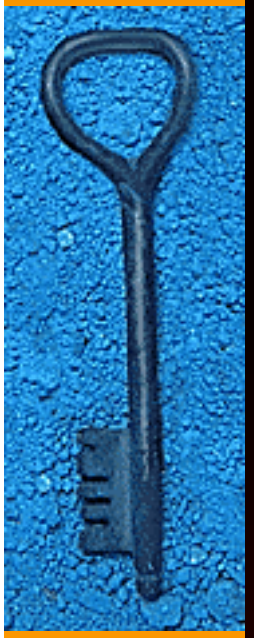
  pack int, $\langle 3, \lambda x.\ x \bmod 2 \rangle$
  as $\exists \alpha.\alpha \times (\alpha \rightarrow \text{int})$

- Using encryption:

  new k in $\langle \{3\}_k, \lambda \{x\}_k.\ x \bmod 2 \rangle$

new k in ...
  generate a fresh key k

$\{v\}_k$
  a value v encrypted by the key k



Secrecy as Non-Interference as Contextual Equivalence

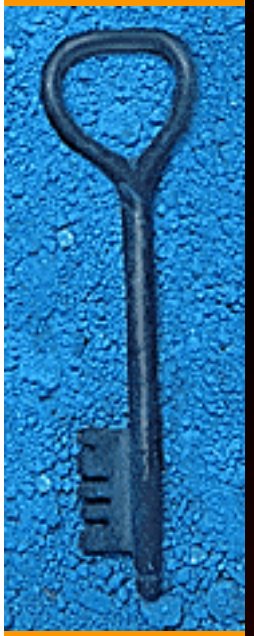
- *Relational parametricity* [Reynolds 83] (or *representation independence* [Mitchell 86]):

$$\text{pack int}, \langle 3, \lambda x.\ x \bmod 2\rangle \text{ as } \exists\alpha.\alpha\times(\alpha\to\text{int})$$

$$\approx \text{pack int}, \langle 1, \lambda x.\ x \bmod 2\rangle \text{ as } \exists\alpha.\alpha\times(\alpha\to\text{int})$$

- This work:

$$\text{new k in } \langle \{3\}_k, \lambda\{x\}_k.\ x \bmod 2\rangle$$
$$\approx \text{new k in } \langle \{1\}_k, \lambda\{x\}_k.\ x \bmod 2\rangle$$



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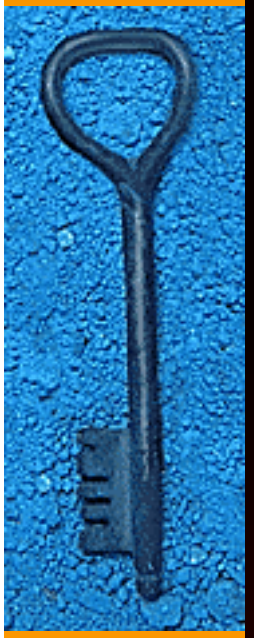


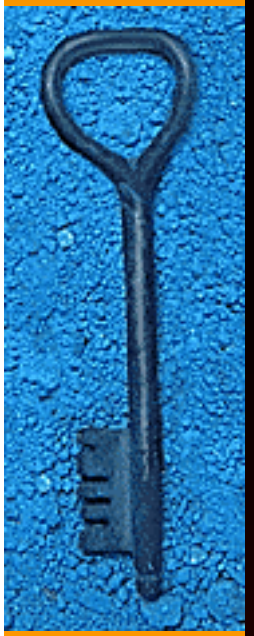
Principle of Parametricity: "Related" Values are Equivalent

$$\varphi \vdash e \sim e' : \tau \quad \text{for some } \varphi$$

$$\Downarrow$$

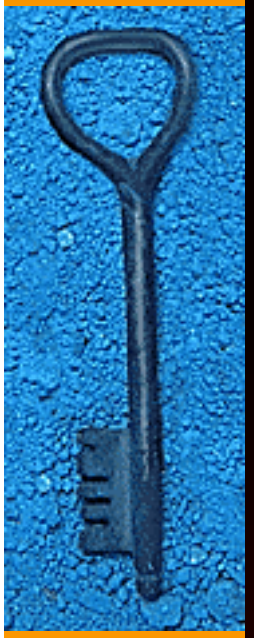
$$[\![f\ e]\!] = [\![f\ e']\!] \quad \text{for any } f : \tau \to \text{bool}$$

- $\sim$ is defined by induction on τ
- φ defines the case of abstract types, mapping each free type variable to a relation between values of its concrete types



Definition of the Logical Relation (1/2)

- Values of a base type (e.g. int) are related iff they are equal
- Functions are related iff they map related arguments to related results
- Pairs are related iff their elements are respectively related



Definition of the Logical Relation (2/2)

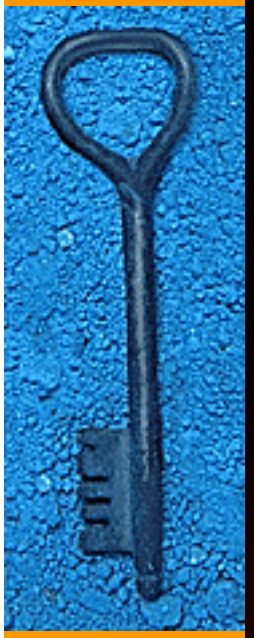
- Packages are related iff their implementations can be related

  $\varphi \vdash \text{pack } \tau,v \text{ as } \exists\alpha.\sigma \sim$
  $\text{pack } \tau',v' \text{ as } \exists\alpha.\sigma : \exists\alpha.\sigma \Leftrightarrow$

  $\varphi, \alpha \mapsto r \vdash v[\tau/\alpha] \sim v'[\tau'/\alpha] : \sigma$
  for some $r \subseteq \tau \times \tau'$

- Values of an abstract type α are related iff they are related by $\varphi(\alpha)$

  $\varphi \vdash v \sim v' : \alpha \Leftrightarrow (v,v') \in \varphi(\alpha)$

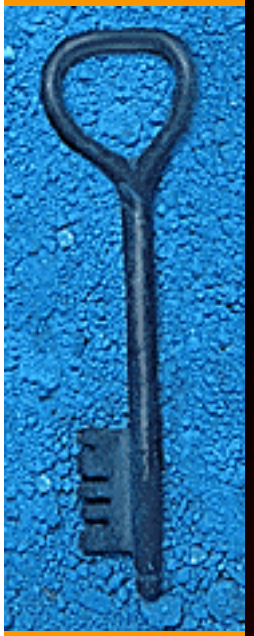


Example

$\alpha \mapsto \{(3,1)\} \vdash$
$\langle 3, \lambda x.\ x \bmod 2\rangle \sim \langle 1, \lambda x.\ x \bmod 2\rangle$
$: \alpha \times (\alpha \to \text{int})$

$\Downarrow$

$\vdash$ pack int, $\langle 3, \lambda x.\ x \bmod 2\rangle$
as $\exists \alpha.\alpha\times(\alpha\to\text{int})$
$\sim$ pack int, $\langle 1, \lambda x.\ x \bmod 2\rangle$
as $\exists \alpha.\alpha\times(\alpha\to\text{int}) : \exists \alpha.\alpha\times(\alpha\to\text{int})$



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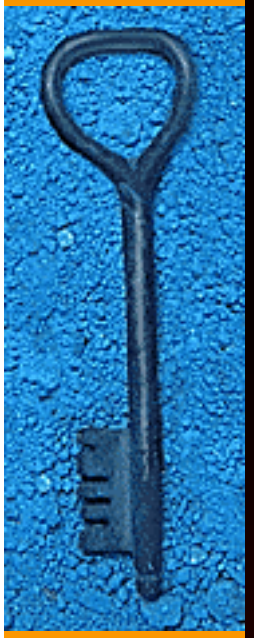
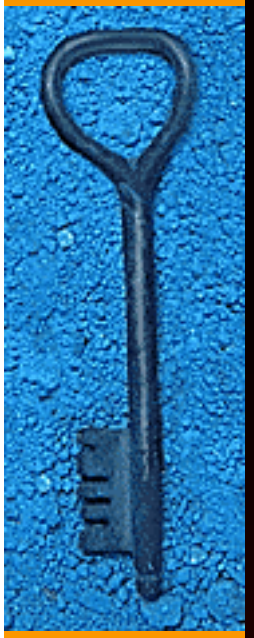


Cryptographic λ-Calculus: Syntax and Semantics

Simply-typed call-by-value λ-calculus + shared-key cryptographic primitives

- e ::= … | k | $\{e1\}_{e2}$ | let $\{x\}_{e1}$ = e2 in e3 else e4
- τ ::= … | key | bits(τ)

let $\{x\}_k$ = $\{v\}_{k'}$ in e else e'
→ e[v/x] if k = k', e' otherwise

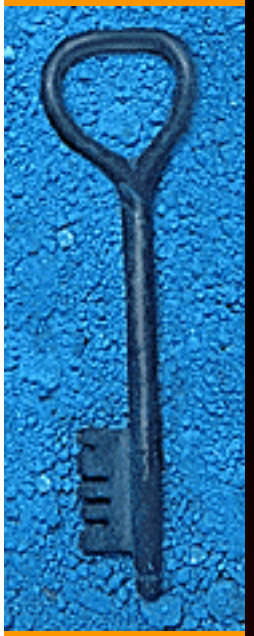


Cryptographic λ-Calculus: Typing Rules

$$\Gamma \vdash k : \text{key}$$

$$\frac{\Gamma \vdash e1 : \tau \qquad \Gamma \vdash e2 : \text{key}}{\Gamma \vdash \{e1\}_{e2} : \text{bits}(\tau)}$$

$$\frac{\Gamma \vdash e1 : \text{key} \qquad \Gamma \vdash e2 : \text{bits}(\tau') \qquad \Gamma, x : \tau' \vdash e3 : \tau \qquad \Gamma \vdash e4 : \tau}{\Gamma \vdash \text{let } \{x\}_{e1} = e2 \text{ in } e3 \text{ else } e4 : \tau}$$



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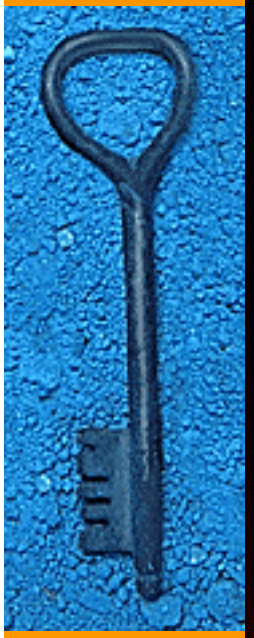


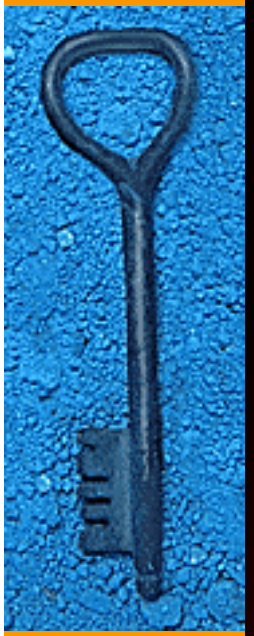
Parametricity Adapted

$$\varphi \vdash e \sim e' : \tau \quad \text{for some } \varphi$$

$$\Downarrow$$

$$[\![f\ e]\!] = [\![f\ e']\!] \quad \text{for any } f : \tau \to \text{bool}$$
$$\text{s.t. } \text{dom}(\varphi) \cap \text{keys}(f) = \varnothing$$

- $\sim$ is defined by induction on τ, meaning that e and e' are equivalent and don't leak the secret keys
- φ defines the case of bits(τ), mapping each secret key to a relation between values encrypted by the key



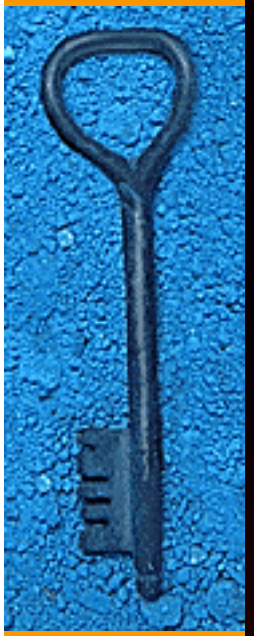
Definition of the Logical Relation

- Keys are related iff they are equal *and non-secret*

  $\varphi \vdash k \sim k : \text{key} \Leftrightarrow k \notin \text{dom}(\varphi)$

- Values encrypted by a secret key k are related iff they are related by $\varphi(k)$

  $\varphi \vdash \{v\}_k \sim \{v'\}_k : \text{bits}(\tau) \Leftrightarrow$
  $(v,v') \in \varphi(k)$ if $k \in \text{dom}(\varphi)$
  $\varphi \vdash v \sim v' : \tau$ if $k \notin \text{dom}(\varphi)$



Example

$k \mapsto \{(3,1)\} \vdash \{3\}_k \sim \{1\}_k : \mathrm{bits(int)}$

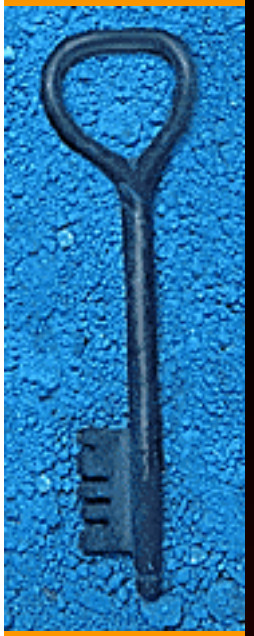
$k \mapsto \{(3,1)\} \vdash \lambda\{x\}_k.\ x \bmod 2$
$\quad \sim \lambda\{x\}_k.\ x \bmod 2 : \mathrm{bits(int)} \to \mathrm{int}$

$\Downarrow$

$k \mapsto \{(3,1)\} \vdash \langle \{3\}_k, \lambda\{x\}_k.\ x \bmod 2 \rangle \sim$
$\langle \{1\}_k, \lambda\{x\}_k.\ x \bmod 2 \rangle$
$\quad : \mathrm{bits(int)} \times (\mathrm{bits(int)} \to \mathrm{int})$

N.B.

$\lambda\{x\}_k.\ e \equiv \lambda z.\ \mathrm{let}\ \{x\}_k = z\ \mathrm{in}\ e\ \mathrm{else}\ \bot$



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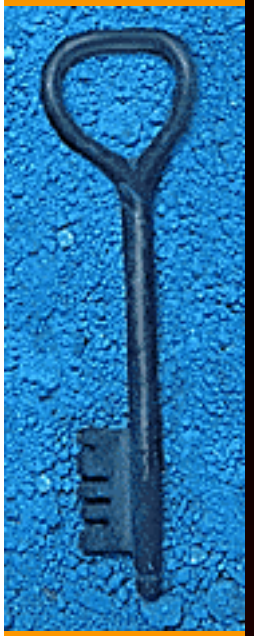
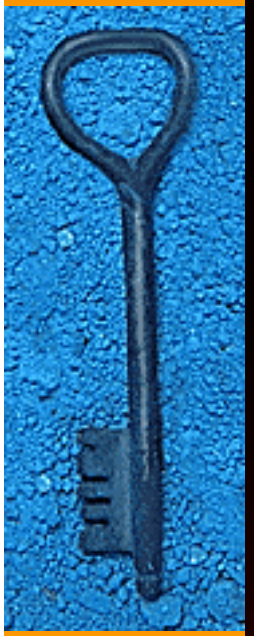


Current Status

- Treatment of fresh key generation, adapting [Stark 97]
- (In)correctness proof of a few security protocols, using the following encoding
  - principal = function from messages to messages (with its own continuation)
  - configuration = record of principals and non-secret keys
  - network and scheduler = "right" context
  - attacker = arbitrary context



Fresh Key Generation (1/2)

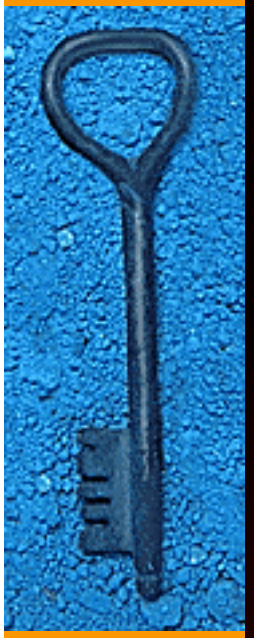
- Syntax

$$e ::= \ldots \mid \text{new } x \text{ in } e$$

- Semantics

$$e \Downarrow (S)v$$

read as: "the expression e evaluates to the value v, generating the set S of fresh keys"; note that (S) is a binder



Fresh Key Generation (2/2)

- Logical relation

  $\varphi \vdash e \sim e' : \tau \;\Leftrightarrow$

  $e \Downarrow (\{k_1,\ldots,k_n\} \oplus S)\ v_1$,

  $e' \Downarrow (\{k_1,\ldots,k_n\} \oplus S')\ v_2$, and

  $\varphi, k_1 \mapsto r_1, \ldots, k_n \mapsto r_n \vdash v_1 \sim v_2 : \tau$

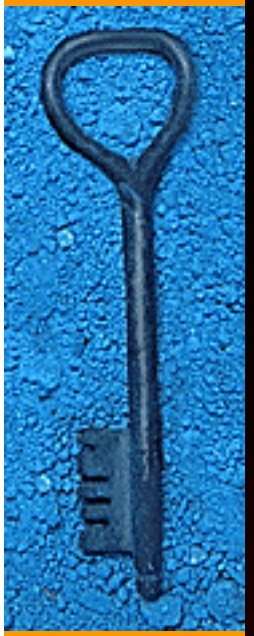
  for some $k_1, \ldots, k_n, r_1, \ldots, r_n$, S and S'

For example,

$\vdash$ new k in $\langle \{3\}_k, \lambda\{x\}_k.\ x \bmod 2 \rangle \sim$

new k in $\langle \{1\}_k, \lambda\{x\}_k.\ x \bmod 2 \rangle$

$: \text{bits(int)} \times (\text{bits(int)} \to \text{int})$



Future Work

- Recursive functions/types

  cf. [Birkedal & Harper 97], [Crary & Harper], etc.

- Concurrency and distribution

  cf. spi-calculus [Abadi & Gordon 97], evaluation semantics for CCS [Pitts 96], typed equivalence in polymorphic π-calculus [Pierce & Sangiorgi 97], parametricity in linear polymorphic λ-calculus [Pitts 2000], etc.